ALGEBRAIC VECTOR BUNDLES OVER COMPLEMENTS OF HYPERPLANE ARRANGEMENTS IN AFFINE SPACES OVER A FIELD

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In questions related to algebraic models of chiral conformal field theory [1], the author encountered the following question: Is every algebraic vector bundle over the ordered configuration space of n points in an affine line over a field trivial? In the present note, we answer the question in the affirmative. In fact, we prove a much more general statement, namely that over any complement of a (finite) arrangement of hyperplanes in an affine space over a field, any algebraic vector bundle is trivial. We use a method of R. Swan [3], which, as mentioned in [3], is originally due to M.P.Murthy. The proof (which is completely elementary) was inspired by conversations with Mike Hopkins on $\mathbb{A}^1$-homotopy theory.

Theorem 1. *Let K be a field, $n = 1, 2, ...$ a natural number and let $L_1, \dots, L_m$ be nonzero polynomials of degree ≤ 1 in variables $x_1, \dots, x_n$. Then every finitely generated projective module over*

$$A = K[x_1, \dots, x_n][L_1^{-1}, \dots, L_m^{-1}]$$

is free. In other words, A is a Quillen-Suslin ring.

Proof. Induction on

$$(n, m) \tag{1}$$

ordered lexicographically from the left (meaning that the leftmost entry has the greatest weight). For $n = 0$, the statement is obvious. Thus, let us assume $n > 1$. We distinguish two cases:

Case 1: $m > 0$. Since clearly any constant polynomials L_i can be dropped, without loss of generality, we may assume that x_n has a nonzero coefficient in L_m, and by linear substitution, we may then assume that

$$L_m = x_n.$$

Let M be a finitely generated projective A-modules. Then the statement is true when A is replaced by the ring

$$\widetilde{A} = K(x_n)[x_1, \dots, x_{n-1}][L_1^{-1}, \dots, L_{m-1}^{-1}]$$

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by the induction hypothesis applied to (1) replaced by

$$(n-1, m-1)$$

and with K replaced by $K(x_n)$. Thus, $M \otimes_A \widetilde{A}$ is a free $\widetilde{A}$-module. By the limit argument, there exists a monic polynomial $f(x)$ in one variable x with coefficients in K such that $M[f(x_n^{-1})^{-1}]$ is a free module over

$$A[f(x_n^{-1})^{-1}]. \tag{2}$$

Let

$$f(x_n^{-1}) = x_n^{-N} + c_{N-1}x_n^{-N+1} + \cdots + c_0 x_n^0,\ c_i \in K.$$

Since $x_n, x_n^{-1} \in A$, (2) is equal to

$$A[g(x_n)^{-1}]$$

where

$$g(x_n) = x_n^N f(x_n^{-1}) = 1 + c_{N-1}x_n + \cdots + c_0 x_n^N.$$

Clearly, $V(g(x_n) = 0)$ is disjoint from the locus $V(x_n = 0)$ in $Spec(B)$ where $V(E)$ denotes the locus of the equation E and

$$B = K[x_1, \ldots, x_n][L_1, \ldots, L_{m-1}].$$

Now thinking of M as a vector bundle over

$$Spec(A) = Spec(B) \smallsetminus V(x_n = 0),$$

the bundle is trivial over

$$Spec(B) \smallsetminus (V(x_n = 0) \amalg V(g(x_n) = 0))$$

and thus may be patched with a trivial bundle over

$$Spec(B) \smallsetminus V(g(x_k) = 0)$$

to define a bundle $\overline{M}$ over $Spec(B)$. By the induction hypothesis, however, our statement is true with A replaced by B (by replacing (n, m) with $(n, m-1)$). Thus, $\overline{M}$ is a trivial bundle, and hence so is M.

Case 2: $m = 0$. In this case, the statement follows from the Quillen-Suslin theorem which solved the Serre conjecture [2]. □

References

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