DERIVED REPRESENTATION THEORY AND STABLE HOMOTOPY CATEGORIFICATION OF sl_k

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Abstract. We set up foundations of representation theory over S, the stable sphere, which is the "initial ring" of stable homotopy theory. In particular, we treat S-Lie algebras and their representations, characters, $gl_n(S)$-Verma modules and their duals, Harish-Chandra pairs and Zuckermann functors. As an application, we construct a Khovanov sl_k-stable homotopy type with a large prime hypothesis, which is a new link invariant, using a stable homotopy analogue of the method of Sussan.

1. Introduction

The primary result of this paper is to set up the foundations, and do some basic computations in representation theory over the sphere spectrum S. The motivation which guided the present investigation originated in knot theory. An important new direction of homotopy theory called *categorification* was started around the year 2000 by Khovanov [16] with a categorification of the Jones polynomial, which became known as Khovanov homology. This invariant is easily defined directly (for a particularly neat definition, see D.Bar-Natan [2]), but actually comes from the categorification of representations of sl_2 (see Bernstein, Frenkel, Khovanov [4]). Here "categorification" means the introduction of a certain categories of chain complexes whose Grothendieck groups, tensored with $\mathbb{C}$, are the representations in question. Morphisms of representations correspond to functors, and as a result, instead of a "knot polynomial", we obtain a chain complex whose homology is Khovanov homology of the given type. Many more categorifications appeared since. From the point of view of the present paper, the most important one is the paper [32] by Joshua Sussan, which gave, by analogy with [4], a categorification of a certain part of the representation theory of sl_k, thus defining what became known as sl_k*-Khovanov homology.*

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A "second revolution" in knot theory, and a major connection between that field and homotopy theory, was started in 2012 in the paper [25] by Robert Lipshitz and Sucharit Sarkar, in which the authors defined an *sl_2-Khovanov stable homotopy type*, using a machinery by Cohen, Jones and Segal, defining a Morse theory stable homotopy type. A different construction, using an abstract approach to topological quantum field theory, was obtained shortly afterward by Hu, I.Kriz and D.Kriz in [11]. Lipshitz and Sarkar [26] also showed that the sl_2-Khovanov stable homotopy type produces non-trivial spectra in the sense of stable homotopy theory, as well as distinguishing pairs of knots which were not distinguished by previously known invariants. The equivalence of the constructions of [25] and [11] was recently proved by T. Lawson, R.Lipshitz and S.Sarkar in [22], see also [23], with further applications to knot theory.

The constructions [25, 11, 22, 23] of the sl_2-Khovanov stable homotopy type all substantially use the fact that the construction of sl_2-Khovanov homology is completely elementary, and uses no non-trivial representation theory. This is due to the fact that the representation theory of sl_2 is very simple. The authors of the present paper set out to define an sl_k-Khovanov stable homotopy type by finding a stable homotopy analogue of J.Sussan's method [32]. This is the main result of the present paper. There are several limitations of this result, which we need to mention. First of all, in this paper, we do not make any knot computations: we felt that the very deep connection between concepts of stable homotopy theory and representation theory we needed to probe were enough for the subject of one paper, and we postponed any knot computations to future work. The other limitation which should be mentioned concerns any knot stable homotopy type, including the sl_2-Khovanov homotopy type. Khovanov homology actually has another grading, making it more precisely a categorification of the representation theory of a quantum group. This structure, which is also established in [32], appears to be lost by lifting to stable homotopy theory. Accordingly, also the present paper only concerns the stable homotopy categorification of sl_k-representations, and not the corresponding quantum group. Finally, and this is perhaps the most material restriction is that we work at a (linearly) large prime with respect to k. This is needed to remove some difficulties specific to modular representation theory, which at present we do not know how to address in stable homotopy theory, although we feel that, of course, the modular story will ultimately be the most interesting. Still, the large prime story is non-trivial to set up, and the case $k = 2$ shows that it is also non-trivial, as there are knots with odd torsion in their Khovanov homology (the smallest one being the $(5,6)$-torus knot).

Finally, it should be mentioned that there are other possible approaches to an sl_k-homotopy type. For example, sl_k-Khovanov homology was also constructed by Khovanov and Rozansky [17, 18] using matrix factorization. Efforts to construct an sl_k-stable homotopy type using this method were made in [14]. Our present program of a construction using representation theory over S took off in 2014 after conversations with Jack Morava. The concept of representation theory over S is of independent interest (see for example Lurie [27]).

To describe what type of mathematics we get into when constructing a Khovanov sl_k-homotopy type, we first describe the approach of Sussan [32], which follows the method of Bernstein, Frenkel and Khovanov [4] for the case of $k = 2$. Their approach is to consider the derived category of the Bernstein-Gelfand-Gelfand (BGG) category [13] of representations of gl_n. Here n is another, typically larger natural number without direct relationship to k. In the context of Khovanov homology, it represents, roughly, the number of strands of a horizontal slice of a link projection into the plane. The main feature of the BGG category is that it is really a category of complex representations of *Harish-Chandra pairs* (H, gl_n) where H is the exponentiation of a Cartan subalgebra, say, consisting of the diagonal matrices. In the context of (derived) BGG, we further restrict to certain finite complexes of special representations called *Verma and co-Verma modules*, which is the right approach from the point of view of categorification, but is actually less important, since it does not change Ext-groups (while passing from representations of Lie algebras to Harish-Chandra pairs does change Ext-groups by "rigidifying" the action of H). To get link invariants, we must actually go somewhat further, considering *parabolic BGG categories* with respect to a parabolic Lie algebra $\mathbf{p} \subseteq gl_n$. These are certain full subcategories of the categories of representations of Harish-Chandra pairs (L, gl_n) where L is an exponentiation of the Levi factor of the parabolic $\mathbf{p}$. The parabolic BGG categories decompose into *blocks*, and taking blocks with certain prescribed weights gives a categorification of tensor products of exterior powers of the standard k-dimensional representation of gl_k, with morphisms of representations categorified by *left and right Zuckermann functors*, which are adjoints to forgetful functors between Harish-Chandra pairs. To get a link complex of a tangle, one takes a *Khovanov cube* of functors associated with a link projection (where, like in the original Khovanov homology, each crossing is a cone, thus responsible for one additional dimension of the cube). If we have a link instead of a tangle, we just get a chain complex, and its cohomology is sl_k-Khovanov homology. (Again, to be completely precise, Sussan [32] works with a quantum group instead of gl_n, thereby producing an additional grading, but this structure is lost in stable homotopy.)

To obtain a stable homotopy version of the construction of [32] we just described, one needs to develop an analogue of the above described $\mathbb{C}$-valued representation theory over S, the sphere spectrum. Of course, even over $\mathbb{Z}$, the representation theory does not work nearly as nicely as over $\mathbb{C}$ (see [20]). We circumvent this difficulty by working over *a large prime*. This means a prime linearly larger than the number k. It is important to note that we cannot assume to be at a prime larger than n, since n depends on the size of the knot or link. While working at a large prime is of course a major and undesirable restriction, it allows non-trivial results over $\mathbb{Z}$, as seen in the examples of odd torsion in Khovanov homology.

Over S, on the other hand, even at a large prime, we face a formidable array of technical and conceptual challenges, and in some sense, resolving them is the main contribution of the present paper. First, we must recall what S is - the stable sphere, which can also be thought of as the "absolute generalized homology theory", or, as a homotopy theorist would say, *the sphere spectrum.* It became clear in the 1990's that one can do algebra over S by developing an analogue of the tensor product of chain complexes which is commutative, associative and unital. The resulting field that began to open up was called by Peter May *brave new algebra*, and later by Jacob Lurie *derived algebraic geometry.*

The difficulty with spectra (i.e., a rigid category of generalized (co) homology theories) is that the geometric model of the shift, which is compatible with the tensor product, is the *suspension*, which involves a choice of an intrinsically non-canonical coordinate. Because of this, the naive approach of the analogue of the tensor product in spectra (which realizes products in (co)homology theories, and was dubbed, by Frank Adams, the *smash product*), lacks strict commutativity and associativity, thereby restricting severely the type of algebraic constructions one could do. This is why a highly technical procedure, developing a symmetric monoidal smash product, is needed before one can do brave new algebra, derived algebraic geometry, or representation theory over S.

One such construction was given in [9], but as is often the case in similar situations, other approaches emerged as well, and their equivalence was later proved in [28].

In this paper, we use the foundations of [9], which have certain technical advantages from our point of view (for cognoscenti, the advantage is that in the Quillen model structure, *every object is fibrant*). The first step toward representation theory is to define Lie algebras over S. That was actually done by M.Ching [6] who constructed a stable homotopy *Lie operad*, using the Goodwillie derivative of the identity. It turns out that the correct concept of a Lie algebra representation is provided by the concept of *operad module*. This requires some thought because an analogous statement is true for some operads and not others: the correct concept of a module over an E_∞-algebra is an operad module, while the operad module for associative algebra is a *bimodule* - a certain modification is needed to get left and right modules, and this is something we must come to terms with in the present paper as well.

Now that we know what a Lie algebra and a representation is, we need examples. Lie algebras, at least from the point of view of what we need to model [32], are not a problem: gl_n, as well as all its parabolics, turn out to be easy, since they arise from a structure of *associative algebras*. The forgetful functor from associative algebras to Lie algebras has an analogue in [6]. (Therefore, we also have its left adjoint, which is the S-analogue of the *universal enveloping algebra*.) Examples of representations are less easy. While we have the "standard" representations, and they are important, to define Verma and co-Verma modules, we also need to model the concept of a *character* of an "abelian" Lie algebra (namely the Cartan algebra of gl_n). An abelian Lie algebra is one which comes not only by forgetting structure from an associative algebra, but from a *commutative* algebra. Classically, it is obvious that the Lie bracket is then 0, which is needed in defining characters.

Over S, the "vanishing of the Lie bracket", in the appropriate operadic sense, for an abelian Lie algebra, is a non-trivial theorem. To prove it, we in fact make another construction of the Lie operad, modeling, over S, the *Koszul dual* of the infinite little cube operad. This brings out another issue: classically, Koszul duality involves *shifts*. The Koszul dual of the little cube operad is its shift. (Unless we shift back, we obtain a graded Lie bracket which itself has a non-zero degree.) In chain complexes, we may, of course, shift at will, but can we do that with operads over S, given that the shift is replaced by the much less well behaved suspension? Luckily, for this, too, a nascent technique was developed by Arone and Kankaanrinta [1]. In this paper, we develop it fully, showing, in fact, that at least on the level of derived categories, operads over S are *stable* in the sense that shifts can be modeled by equivalences in categories.

Having constructed characters of abelian Lie algebras over S, we now have Verma and co-Verma modules. Unrestricted groups of derived morphisms of Verma and co-Verma modules over S, however, are quite "wild", involving Spanier-Whitehead duals of very infinite spectra, and we do not have computations. The situation is somewhat better in a variant of our construction, which is *graded by weights*. There, some computations are possible using Carlsson's solution [5] of the Segal conjecture. Still, those are not the groups we need for knot theory.

Enter Harish-Chandra pairs. This concept of pairs of a Lie algebra, and compatible group in some sense (Lie or algebraic), occurs in several areas of representation theory. But how do we model Harish-Chandra pairs over S? We do not have Lie groups over S, but it turns out that we do have algebraic groups, at least in a suitably weak sense: We can define a meaningful concept of a commutative Hopf algebra, and we can invert a homotopy class. From this point of view, we have a model of $\mathscr{O}_{GL_nS}$. To have Harish-Chandra pairs, however, we need to model a co-action of such "S-algebraic groups" on a Lie algebra. This is, at present, a problem, since we do not know of a way of rigidifying sufficiently the Hopf algebra *conjugation*, which, for GL_n, models the inverse matrix. We have a workaround with a large prime hypothesis (which is OK, since the n here is actually not the n mentioned earlier - we only need Harish-Chandra pairs where the group is a product of GL_ℓ's for $\ell \leq k$).

Once the large prime hypothesis is adopted, the theory of Harish-Chandra pairs, at least on blocks of limited weights, behaves well, as one may basically refer to characteristic 0 for calculations. We managed to define analogues of Zuckermann functors, model a stable homotopy analogue of the Khovanov cube, and also formalize Sussan's *diagram relations* to a point where they can be reduced to existence of dualizing objects, which, again, follows from the characteristic 0 case. Thus, we have constructed an sl_k Khovanov homology stable homotopy type, at a prime (linearly) large with respect to k, and prove its link invariance.

The present paper is organized as follows: In Section 2, we introduce the stable homotopy foundations we work with. In Section 3, we introduce S-Lie algebras and their representations. In Section 4, we focus on the example of $gl_n S$, and define Verma modules and some other examples of interest. In Section 5, we define Harish-Chandra pairs and Zuckermann functors. In Section 6, we apply this to the construction of the sl_k Khovanov homotopy type, and its link invariance.

2. Operads in based spaces and spectra

2.1. **Basic setup and model structure.** Let $\mathscr{B}$ be the category of based spaces with the usual smash product $\wedge$ with the usual model structure where equivalences are weak equivalences and cofibrations are retracts of relative cell complexes. We need a category $\mathscr{S}$ of *spectra* with the following properties:

(1) There is a small object model structure for which every object is fibrant, and a suspension spectrum functor $\Sigma^\infty : \mathscr{B} \to \mathscr{S}$ which is a left Quillen adjoint. (We will often tend to omit this functor from the notation.) Further, there is a notion of geometric homotopy of morphisms such that cofibrant objects are co-local with respect to the geometric homotopy category (i.e. objects and geometric homotopy classes of morphisms).

(2) There is a commutative, associative and unital smash product $\wedge$ in $\mathscr{S}$ such that the suspension spectrum functor preserves the smash product.

One example of such a category $\mathscr{S}$ is constructed in [9]. The construction goes as follows: Let $\mathcal{S}$ denote the category of May spectra over a given universe. Denoting by $\mathscr{L}$ the linear isometries operad on a given universe, one first constructs the category $\mathcal{S}[\mathscr{L}]$ of $\mathscr{L}(1)$-modules, whose objects are spectra X together with an action

$$\mathscr{L}(1) \ltimes X \to X,$$

and morphisms are morphisms of spectra which preserve this structure. Denoting by $L(1)$ the monad defining $\mathscr{L}(1)$-modules, we can consider the monad $(L(1), L(1))$ on pairs of spectra. Then there is a right $(L(1), L(1))$-module $L(2)$ where

$$L(2)(X, Y) = \mathscr{L}(2) \ltimes (X \wedge Y)$$

(where on the right hand side, $\wedge$ denotes the external smash product). We then define the smash product in $\mathcal{S}[\mathscr{L}]$ as

$$X \wedge_{\mathscr{L}} Y = L(2) \otimes_{(L(1),L(1))} (X, Y)$$

where $\otimes_{(L(1),L(1))}$ denotes the coend. This operation is commutative and associative but not strictly unital. It is important, however, that

S, the suspension spectrum of S^0, is a commutative semigroup with respect to the operation $\wedge_{\mathscr{L}}$ in $\mathcal{S}[\mathscr{L}]$.

The model structure on $\mathcal{S}[\mathscr{L}]$ has the property that fibrations resp. equivalences are those morphisms which forget to fibrations resp. equivalences in spectra. The forgetful functor from $\mathcal{S}[\mathscr{L}]$ to $\mathcal{S}$ is then right Quillen adjoint to $L(1)$ (thought of as a functor from spectra to $\mathcal{S}[\mathscr{L}]$). The desired suspension spectrum functor is the composition of the May suspension spectrum functor with $L(1)$.

As remarked, the smash product $\wedge_{\mathscr{L}}$ is not strictly unital, although

$$S \wedge_{\mathscr{L}} S \cong S.$$

This can be remedied by the following further trick: Denote by $\mathscr{S}$ the full subcategory of $\mathcal{S}[\mathscr{L}]$ on objects X for which the canonical morphism

$$S \wedge_{\mathscr{L}} X \to X$$

is an isomorphism. Then we have a functor

$$S \wedge_{\mathscr{L}} (?) : \mathcal{S}[\mathscr{L}] \to \mathscr{S}, \tag{1}$$

left adjoint to the functor

$$F_{\mathscr{L}}(S, ?) : \mathscr{S} \to \mathcal{S}[\mathscr{L}]. \tag{2}$$

(Both functors are restrictions of self-functors on $\mathcal{S}[\mathscr{L}]$. It is proved in [9] that there is a model structure on $\mathscr{S}$ such that the pair of adjoint functors (1), (2) is a Quillen equivalence.

We will work in the category $\mathscr{S}$. From now on, we will abuse terminology by referring to $\mathscr{S}$ as the category of *spectra*, and denoting the smash product $?\wedge_{\mathscr{L}}?$ restricted to $\mathscr{S}$ simply by $\wedge$.

Definition 1. *An* operad *in $\mathscr{B}$ is a sequence of objects $\mathscr{C}(n)$, $n = 0, 1, 2, \ldots$, and morphisms*

$$\gamma : \mathscr{C}(k) \wedge \mathscr{C}(n_1) \wedge \cdots \wedge \mathscr{C}(n_k) \to \mathscr{C}(n_1 + \cdots + n_k),$$

$$\iota : S^0 \to \mathscr{C}(1)$$

and actions of Σ_k on $\mathscr{C}(k)$ which satisfy the usual associativity, unit and equivariance axioms [29]*. An operad in $\mathscr{S}$ is defined the same way except that S^0 is replaced by S. A* 1-operad *is defined the same way, except that we have $n = 1, 2, \ldots$.*

An algebra *over an operad $\mathscr{C}$ in $\mathscr{B}$ or $\mathscr{S}$ is an object X together with structure maps*

$$\mathscr{C}(n) \wedge \underbrace{X \wedge \cdots \wedge X}_{n} \to X$$

which satisfy the usual associativity, unit and equivariance axioms ([29]). For an algebra R over an operad (or 1-operad) $\mathscr{C}$), a $(\mathscr{C}, R)$-module *M has structure maps*

$$\mathscr{C}(n) \wedge \underbrace{X \wedge \cdots \wedge X}_{n-1} \wedge M \to M$$

which satisfy the usual associativity, equivariance and unitality axioms (see e.g. [9]).

Morphisms of operads, 1-operads, algebras and modules are (sequences of) morphisms in the underlying category which preserve the operations.

The category of 1-operads is (canonically equivalent to) a coreflexive subcategory of the category of operads. The inclusion functor takes a 1-operad $\mathscr{C}$ to the operad where we additionally define $\mathscr{C}(0) = *$ (the zero object). The coreflection functor (right adjoint to the inclusion) is given on an operad $\mathscr{C}$ by forgetting $\mathscr{C}(0)$. The category of algebras over a 1-operad $\mathscr{C}$ is canonically equivalent to the category of algebras over $\mathscr{C}$ considered as an operad, and similarly for modules.

The category of associative unital algebras (in $\mathscr{B}$ or $\mathscr{S}$) is additionally a reflexive subcategory of the category of 1-operads. The inclusion functor is defined, on an associative algebra $\mathscr{A}$, by putting $\mathscr{A}(1) = \mathscr{A}$, $\mathscr{A}(n) = *$ for $n \geq 2$. The reflection (left adjoint to inclusion) is defined by sending an operad $\mathscr{C}$ to the associative algebra $\mathscr{C}(1)$.

The category of operads (resp. 1-operads) has a terminal object $*$ where $*(n) = *$ for all n, and an initial object $\mathscr{E}$ where $\mathscr{E}(1) = S^0$ and $\mathscr{E}(n) = *$ for $n \neq 1$. The category of $\mathscr{E}$-algebras is canonically equivalent to the underlying category ($\mathscr{B}$ or $\mathscr{S}$).

Additionally, Σ^∞ extends to a functor

$$\Sigma^\infty : \mathscr{B}\text{-operads} \to \mathscr{S}\text{-operads}.$$

On $\mathscr{B}$ and $\mathscr{S}$, there are model structures where cofibrations are retracts of relative cell complexes where cells are of the form

$$\begin{array}{ll} S^{n-1}_+ \to D^n_+,\ n \geq 0 & \text{in } \mathscr{B} \\ S^{n-1}_+ \to D^n_+,\ n \in \mathbb{Z} & \text{in } \mathscr{S}, \end{array} \tag{3}$$

equivalences are maps inducing an isomorphism in π_*. Then Σ^∞ is a left Quillen adjoint. Note also that the smash product of two cofibrant objects in $\mathscr{B}$ or $\mathscr{S}$ is cofibrant.

Similar comments also apply to 1-operads, and the pairs of adjunct functors between 1-operads and operads, and 1-operads and associative algebras, are Quillen adjunctions, as usual.

The reason we have to consider both operads and 1-operads is subtle. Concepts such as stability and Koszul duality work better for 1-operads. On the other hand, the tensor product of modules over an operad algebra works better for operads.

Now we have forgetful functors

$$\begin{aligned} ?(n) &: \mathscr{B}\text{-operads} \to \mathscr{B} \\ ?(n) &: \mathscr{S}\text{-operads} \to \mathscr{S} \end{aligned} \tag{4}$$

for $n \in \mathbb{N}_0$, with left adjoint $\mathscr{O}(n)$. Then we have model structures on $\mathscr{B}$-operads and $\mathscr{S}$-operads where cofibrations are relative cell complexes where cells are $\mathscr{O}(n)$ of cells of the form (3), and equivalences are morphisms of operads which become equivalences upon applying (4) for every n. Then Σ^∞ extends to a left Quillen adjoint

$$\mathscr{B}\text{-operads} \to \mathscr{S}\text{-operads}.$$

For the purposes of homotopy theory, we will only consider algebras over cofibrant operads. For a cofibrant operad $\mathscr{C}$ (in $\mathscr{B}$ or $\mathscr{S}$), there is a model structure on $\mathscr{C}$-algebras where equivalences and fibrations are the morphisms which have the same property in the underlying category. The functor Σ^∞ then takes $\mathscr{C}$-algebras to $\Sigma^\infty\mathscr{C}$-algebras, and is a left Quillen adjoint.

Suppose $f : \mathscr{C} \to \mathscr{D}$ is a morphism of operads in $\mathscr{B}$ or $\mathscr{S}$, where $\mathscr{C}$ and $\mathscr{D}$ are cofibrant. Then every $\mathscr{D}$-algebra X is automatically a $\mathscr{C}$-algebra. We will denote this functor from $\mathscr{D}$-algebras to $\mathscr{C}$-algebras by f^*. It is a right Quillen adjoint to a functor which we will denote by $f_\sharp$.

2.2. **Stability.** In the category of chain complexes, there is a suspension functor $?[1]$ on 1-operads given by

$$\mathscr{C}[1](n) = \mathscr{C}(n)[n-1]$$

where the brackets on the right hand side mean ordinary suspension in the category of chain complexes:

$$C[k]_n = C_{n-k}.$$

For an 1-operad $\mathscr{C}$, there is then an equivalence of categories

$$\mathscr{C}\text{-algebras} \to \mathscr{C}[1]\text{-algebras},$$

$$X \mapsto X[-1].$$

We shall prove an analogous result for spectra, at least on the level of derived (=Quillen homotopy) categories of algebras over cofibrant 1-operads.

The Arone-Kankaanrantha (AK) 1-operad [1] in $\mathscr{B}$ has

$$\mathscr{J}(n) = \Delta^{n-1}/\partial\Delta^{n-1}$$

where Δ^{n-1} is the standard $(n-1)$-simplex. The barycentric coordinates are written as $[t_1,\dots,t_n]$, the symmetric group Σ_n acts by permutation of coordinates, and composition is given by

$$\begin{array}{l}[s_1,\dots,s_k]\times[t_{1,1},\dots,t_{1,n_1}]\times\dots\times[t_{k,1},\dots,t_{k,n_k}]\\ \mapsto[s_1t_{1,1},\dots,s_1t_{1,n_1},\dots,s_kt_{k,1},\dots s_kt_{k,n_k}].\end{array}$$

It is also a co-1-operad, with structure map

$$\mathscr{J}(n_1+\dots+n_k)\to\mathscr{J}(k)\wedge\mathscr{J}(n_1)\wedge\dots\wedge\mathscr{J}(n_k)$$

given by

$$\begin{array}{l}[s_{1,1},\dots,s_{1,n_1},\dots,s_{k,1},\dots,s_{k,n_k}]\mapsto\\ [s_{1,1}+\dots+s_{1,n_1},\dots,s_{k,1}+\dots+s_{k,n_k}]\times\\ \left[\dfrac{s_{1,1}}{s_{1,1}+\dots+s_{1,n_1}},\dots,\dfrac{s_{1,n_1}}{s_{1,1}+\dots+s_{1,n_1}}\right]\times\\ \dots\times\left[\dfrac{s_{k,1}}{s_{k,1}+\dots+s_{k,n_k}},\dots,\dfrac{s_{k,n_k}}{s_{k,1}+\dots+s_{k,n_k}}\right].\end{array}$$

This means that

$$\mathscr{J}=\Sigma^\infty\mathscr{J},$$
$$\mathscr{T}=F(\Sigma^\infty\mathscr{J},S^0)$$

are operads in $\mathscr{S}$. Let $\widetilde{\mathscr{J}}$, $\widetilde{\mathscr{T}}$ be their cofibrant replacements.

Definition 2. *An* E_∞-operad *(resp.* E_∞-1-operad*) is a cofibrant operad (res.* 1*-operad) in* $\mathscr{S}$ *equivalent to* $\Sigma^\infty S^0$*. (Here* S^0 *stands for the unique operad (resp.* E_∞*-*1*-operad) in* $\mathscr{B}$ *whose n'th term is* S^0*.) An algebra over an* E_∞*-operad will be also called an* E_∞*-algebra.*

Lemma 3. $\widetilde{\mathscr{J}}\wedge\widetilde{\mathscr{T}}$ *is an* E_∞*-*1*-operad. (The smash product is performed one term at a time.)*

Proof. For any co-1-operad $\mathscr{R}$ in $\mathscr{B}$, this is true, in $\mathscr{S}$, for $\widetilde{\mathscr{J}}$ and $F(\mathscr{R},S^0)$, provided we have a commutative diagram of the form
(5)

$$\begin{tikzcd}\mathscr{J}(k)\wedge\mathscr{J}(n_1)\wedge\dots\wedge\mathscr{J}(n_k) \arrow[r] \arrow[d] & \mathscr{R}(k)\wedge\mathscr{R}(n_1)\wedge\mathscr{R}(n_1)\wedge\dots\wedge\mathscr{R}(n_k)\\ \mathscr{J}(n) \arrow[r] & \mathscr{R}(n). \arrow[u]\end{tikzcd}$$

However, this is easily verified by definition for $\mathscr{R} = \mathscr{J}$, with the lower horizontal map the identity:

$$[s_1, \dots, s_k] \times [t_{1,1}, \dots, t_{1,n_1}] \times \dots \times [t_{k,1}, \dots, t_{k,n_k}]$$

is mapped to

$$[s_1 t_{1,1}, \dots, s_1 t_{1,n_1}, \dots, s_k t_{k,1}, \dots, s_k t_{k,n_k}],$$

and by the right vertical arrow back to itself. $\square$

Lemma 4. *$\Sigma^\infty S^1$ and the cofibrant replacement $F(\widetilde{\Sigma^\infty S^1}, S^0)$ of $F(\Sigma^\infty S^1, S^0)$ are algebras over $\widetilde{\mathscr{J}}$, $\widetilde{\mathscr{T}}$, respectively, and*

$$\Sigma^\infty S^1 \wedge F(\widetilde{\Sigma^\infty S^1}, S^0) \tag{6}$$

is equivalent to the E_∞-algebra S^0.

Proof. Consider an 1-operad $\mathscr{J}$ and a co-1-operad $\mathscr{R}$ satisfying (5). Purely on the level of spaces, for a space X, for $\Sigma^\infty X$ and $F(\widetilde{\Sigma^\infty X}, S)$ to have the structure described, along with a map of algebras

$$ev : \Sigma^\infty X \wedge F(\widetilde{\Sigma^\infty X}, S) \to S, \tag{7}$$

it suffices to have an $\mathscr{R}$-algebra structure and an $\mathscr{S}$-coalgebra structure on X together with a commutative diagram

$$\begin{tikzcd} X \wedge \dots \wedge X \arrow[r] \arrow[ddr, "Id"'] & X \wedge \mathscr{J}(n) \arrow[d] \\ & X \wedge \mathscr{R}(n) \arrow[d] \\ & X \wedge \dots \wedge X. \end{tikzcd} \tag{8}$$

For $\mathscr{J}$ and $\mathscr{R}$ as above, the formulas

$$\begin{gathered} S^1 \wedge \dots \wedge S^1 \to S^1 \wedge \mathscr{R}(m) \\ (t_1, \dots, t_m) \mapsto (t_1, \dots, t_m) \times \left[\frac{t_1}{t_1 + \dots + t_m}, \dots, \frac{t_m}{t_1 + \dots + t_m} \right] \end{gathered} \tag{9}$$

and

$$\begin{gathered} S^1 \wedge \mathscr{J}(n) \to S^1 \wedge \dots \wedge S^1 \\ t \times [s_1, \dots, s_m] \mapsto (s_1 t, \dots, s_m t) \end{gathered} \tag{10}$$

where $S^1 = [0,1]/0 \sim 1$ would satisfy (8), but unfortunately (10) is somewhat incorrect in that $t = 1$ in (10) does not necessarily imply that the element goes to the base point. Note that (10) is a partial structure in the sense that it is correct if we let the model of S^1 on the left be $[0,N]/0 \sim N$ where $N \geq m$.

A better approach however is replacing S^1 with $\underline{S}^1 = [0,1]\times(0,1]/(0,\epsilon) \sim *, (s,\epsilon) \sim *$ where $s \geq \epsilon$. Then the coalgebra structure can be replaced by a coalgebra structure

$$\underline{S}^1 \wedge \mathscr{J}(n) \wedge \mathscr{C}(n) \to \underline{S}^1 \wedge \cdots \wedge \underline{S}^1$$

for a suitable E_∞-1-operad $\mathscr{C}$ by "decreasing the coordinate ϵ". The action (9) can then be made to "non-increase". The diagram (8) will be commutative up to E_∞-homotopies. Since we use cofibrant replacement everywhere, this is sufficient. We omit the details. □

2.3. **Equivalence of AK-1-operads.** Consider an 1-operad Q on $\mathscr{B}$ given by

$$Q(n) = \overline{A}_n/\overline{B}_n,$$

where

$$\overline{A}_n = \{(J_1,\dots,J_n) \mid J_i = [s_i,t_i],\ 0 \leq s_i < t_i \leq 1\},$$
$$\overline{B}_n = \{(J_1,\dots,J_n) \in \overline{A}_n \mid \mathrm{Interior}(J_1 \cap \cdots \cap J_n) = \varnothing\}.$$

Here $\overline{A}_n$, $\overline{B}_n$ are given the induced topology from $\mathbb{R}^{2n}$. Note that $\overline{B}_n$ is a closed subset of $\overline{A}_n$. Then Q is a 1-operad where composition is given by increasing linear homeomorphisms

$$I \xrightarrow{\cong} J_i.$$

It is easily checked that $Q(n)$ has the same homotopy type as $\mathscr{J}(n)$ where $\mathscr{J}$ is the 1-operad defined in the last section. However, we will need a stronger statement, whose proof is more difficult:

Proposition 5. *There is an equivalence of $\mathscr{B}$-1-operads*

$$Q \sim \mathscr{J}.$$

The proof of the Proposition will occupy the remainder of this subsection. Put

$$A_n = \{(J_1,\dots,J_n) \in \overline{A}_n \mid J_1 \cap \cdots \cap J_n \neq \varnothing\},$$

$$B_n = A_n \cap \overline{B}_n.$$

Then the canonical continuous map

$$A_n/B_n \to \overline{A}_n/\overline{B}_n$$

is a bijection and is in fact a homeomorphism, as $A_n \subset \overline{A}_n$ is a closed subset.

To prove the Proposition, we shall introduce an "intermediate" 1-operad Q'. We let
$$Q'_n = A'_n / B'_n$$
where
$$A'_n = A_n \cup C_n,$$
$$C_n = \{([0,t_1],\dots,[0,t_n] \mid 0 \le t_i \le 1,\ (t_1,\dots,t_n) \ne (0,\dots,0),\ (\exists i) t_i = 0\},$$
$$B'_n = B_n \cup C_n.$$
Again, $A'_n, B'_n \subset \mathbb{R}^{2n}$ are given the induced topology.

It is easy to see that $C_n \subseteq B'_n \subseteq A'_n$ is a closed subset (of course, $C_n \cap A_n = \varnothing$,) and Q' is a 1-operad with structure defined by the same formula as for Q. Furthermore, we have a canonical continuous map of 1-operads
$$\iota : Q \to Q' \tag{11}$$
which is a bijection but not necessarily a homeomorphism.

Lemma 6. *ι is a weak equivalence.*

Proof. We first remark that one easily checks that A_n, A'_n are contractible, and the inclusions $B_n \subset A_n$, $B'_n \subset A'_n$ are cofibrations. Therefore, it suffices to prove that
$$B_n \subset B'_n \text{ is an equivalence.} \tag{12}$$
To this end, we define a map
$$p : B'_n \to [0,1)$$
by putting
$$(J_1,\dots,J_n) \in B_n \mapsto J_1 \cap \dots \cap J_n,$$
$$([0,t_1],\dots,[0,t_n]) \in C_n \mapsto 0.$$
Clearly, p is continuous. Note that (12) would follow from
$$p \text{ is a quasifibration.} \tag{13}$$
Thus, it suffices to prove (13). We will use the criterion of May [29], Theorem 2.6. We remark that $\{0\}$, $(0,1)$ are clearly distinguished sets, as $p : B_n \to (0,1)$ is clearly a fiber bundle (hence, in fact, a product).

We are therefore done if we can exhibit a continuous homotopy
$$\widetilde{\lambda} : B'_n \to B_n$$
which covers multiplication by $\lambda \in [0,1]$ on $[0,1)$ such that $\widetilde{0}$ is an equivalence on fibers. We let
$$\widetilde{\lambda}|_{C_n} = Id,$$

and for

$$J_1 \cap \cdots \cap J_n = \{n\},\ J_i = [s_i, t_i],$$

we have

$$\widetilde{\lambda}(J_1, \ldots, J_n) = ([\phi_{\lambda,u} s_1, \phi_{\lambda,u} t_1], \ldots, [\phi_{\lambda,u} s_n, \phi_{\lambda,u} t_n])$$

where

$$\phi_{\lambda,u} : I \to I$$

is a continuous map preserving 0, 1 such that

$$\phi_{\lambda,u}(u) = \lambda u$$

and $\phi_{\lambda,u}$ is linear on $[0,u]$ and $[u,1]$. Clearly,

$$\widetilde{?} : B'_n \times [0,1] \to B'_n$$

is continuous, and $\widetilde{\lambda}$ covers $\lambda : [0,1] \to [0,1]$. Thus, it remains to show that

$$\widetilde{0} : p^{-1}(u) \to C_n \tag{14}$$

is an equivalence. To this end, define a linear increasing homeomorphism

$$q_u : [u,1] \to [0,1]$$

and a linear *decreasing* homeomorphism

$$r_u : [0,u] \to [0,1].$$

Define

$$\Phi_u : p^{-1}(u) \to I^{2n}$$

by

$$\Phi_u([s_1,t_1], \ldots, [s_n,t_n]) =$$
$$(r_u(s_1), q_u(t_1), \ldots, r_u(s_n), q_u(t_n)).$$

We then see that Φ_u is a homeomorphism onto the set S of all points

$$(x_1, y_1, \ldots, x_n, y_n) \in I^{2n}$$

where

$$(x_i, y_i) \neq (0,0)$$

and

$$(\exists i,j)\ x_i = 0, y_j = 0.$$

Clearly, $S_0 \subset S$ is a homotopy equivalence where

$$S_0 = \{(x_1, y_1, \ldots, x_n, y_n) \in S \mid x_i + y_i = 1\}.$$

Then

$$(x_1, y_1, \ldots, x_n, y_n) \mapsto (x_1, \ldots, x_n)$$

is a homeomorphism

$$\Psi : S_0 \xrightarrow{\cong} R$$

where
$$R = \{(x_1, \dots, x_n) \in I^n \mid (\exists i, j)\ x_i = 0,\ x_j = 1\}.$$
Note that in the canonical decomposition of I^n into cubes, R is the union of open cubes in ∂I^n whose closures do not intersect $(0, \dots, 0)$ and $(1, \dots, 1)$. Hence, we have $R \cong S^{n-2}$.

On the other hand, we have a homotopy equivalence
$$\Theta : C \xrightarrow{\simeq} R,$$
$$([0, t_1], \dots, [0, t_n]) \mapsto (\frac{t_1}{\max(t_i)}, \dots, \frac{t_n}{\max(t_i)}),$$
and we see that
$$\Theta \circ \widetilde{0} \circ \Phi_u^{-1} \circ \Psi^{-1} \simeq Id_R$$
(via a linear homotopy). □

Corollary 7. *(of the proof) The inclusion $C_n \to B'_n$ is an equivalence.*

□

Proof of Proposition 5: We just proved that the inclusion

$$Q \xrightarrow{\subset} Q' \tag{15}$$

is an equivalence of 1-operads. Now note that we have another 1-operad $\mathscr{J}'$ with
$$\mathscr{J}'(n) = C_n$$
and the same composition formula. Thus, by Corollary 7, the inclusion

$$\mathscr{J}' \xrightarrow{\subset} Q' \tag{16}$$

is an equivalence of 1-operads. Finally, we have an obvious inclusion of 1-operads

$$\mathscr{J} \xrightarrow{\subset} \mathscr{J}' \tag{17}$$

which on $\mathscr{J}(n)$ is
$$[t_1, \dots, t_n] \mapsto ([0, t_1], \dots, [0, t_n]).$$
Clearly, this is an equivalence, so by the equivalences (15), (16), (17), we are done. □

3. S-Lie algebras and their representations

3.1. **The derived Lie operad.** Now consider the k-dimensional little cube operad $\mathscr{C}_{k+}$ in $\mathscr{B}$ (where the subscript '$?_+$' denotes a disjoint base point), with n'th space

$$\mathscr{C}_k(n)_+.$$

We know [29] that

$$\mathscr{C}_+ = \lim_{\to} \mathscr{C}_{k+}$$

is an E_∞-operad. Now note that there is a natural map of 1-operads

$$\mathscr{C}_{(k+1)+} \to Q \wedge (\mathscr{C}_{k+}) \tag{18}$$

given by smashing the projection of little cubes to the first coordinate with the projection to the last k coordinates (term-wise from the point of view of the 1-operad). By Lemma 3, we then have maps of 1-operads

$$\widetilde{\mathscr{T}} \wedge \mathscr{C}_{(k+1)+} \to \mathscr{C}_{k+}. \tag{19}$$

To simplify notation, we shall from now on omit the tilde from $\widetilde{\mathscr{T}}$, and write simply $\mathscr{T}$. We shall, however, always mean the cofibrant replacement. Let

$$\mathscr{L} = \operatorname{ho}\lim_{\leftarrow} \mathscr{T}^k \wedge \mathscr{C}_{(k+1)+}, \tag{20}$$

using the maps (19). We shall call $\mathscr{L}$ the *derived Lie* 1*-operad*. We shall also consider $\mathscr{L}$ as an operad by applying the inclusion functor mentioned after Definition 1.

Proposition 8. *We have a diagram of operads in $\mathscr{S}$:*

$$\begin{tikzcd} \mathscr{L} \arrow[r, "u"] \arrow[d, "p"'] & \mathscr{C}_{1+} \arrow[d, "q"] \\ \mathscr{E} \arrow[r, "i"'] & \mathscr{C}_+ \end{tikzcd} \tag{21}$$

which is commutative in the derived (=Quillen homotopy) category.

Proof. It is easy to see explicitly that $\mathscr{L}(1) \simeq S^0$ (since the stabilization is an isomorphism on that term), so p is obtained as the counit of the adjunction between associative algebras and 1-operads (see the comments after Definition 1).

The top map follows directly from the definition (20). Now let us investigate the composition map

$$\mathscr{T}^k \wedge \mathscr{C}_{(k+1)+} \to \mathscr{T}^{k-1} \wedge \mathscr{C}_{k+} \to \ldots \to \mathscr{C}_{1+} \to \ldots \to \mathscr{C}_{(k+1)+}. \tag{22}$$

Smashing with $\mathscr{J}^k$ (again, we omit the tilde, but mean a cofibrant model), we have, in the homotopy category of operads, a map

$$\mathscr{C}_{(k+1)+} \to Q^k \wedge \mathscr{C}_{(k+1)+}$$

which is, more or less by the definition of (18), $\epsilon \wedge Id$ where

$$\epsilon : S^0 \to Q^k$$

is given as follows: Choose a model $\widetilde{S}^0$ where

$$\widetilde{S}^0(n) = \Delta^n_+$$

with operad structure

$$\begin{array}{l}[t_1, \dots, t_k] \times [s_{1,1}, \dots, s_{1,n_1}] \times \dots \times [s_{k,1}, \dots, s_{k,n_k}] \\ \mapsto [t_1 s_{1,1}, \dots, t_1 s_{1,n_1}, \dots, t_k s_{k,1}, \dots, t_k s_{k,n_k}].\end{array}$$

A map of operads $\widetilde{S}^0 \to \mathscr{J}$ is given by the projections

$$\Delta^n_+ \to \Delta^n / \partial \Delta^n.$$

As before, this can be modeled by a map

$$\epsilon : \mathscr{D}_+ \to Q$$

where

$$\mathscr{D}(n)_+ = \{(J_1, \dots, J_n) \mid J_1, \dots, J_n \text{ are closed subintervals of } I\}$$

and ϵ is the projection. We conclude that (22) factors, up to homotopy, through a map of operads

$$\begin{tikzcd}
\mathscr{T}^k \wedge \mathscr{C}_{(k+1)+} \arrow[r] \arrow[d, "\epsilon^k"'] & \mathscr{C}_{(k+1)+} \\
\mathscr{J}^k \wedge \mathscr{T}^k \wedge \mathscr{C}_{(k+1)+}. \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

Additionally, we claim there is a homotopy commutative diagram

$$\begin{tikzcd}
\mathscr{J}^{k-1} \wedge \mathscr{T}^k \wedge \mathscr{C}_{(k+1)+} \arrow[r] \arrow[d, "\epsilon \wedge Id"'] & \mathscr{J}^{k-1} \wedge \mathscr{T}^{k-1} \wedge \mathscr{C}_{k+} \arrow[r, "\simeq"] & \mathscr{C}_{k+} \arrow[d, "\subset"] \\
\mathscr{J}^k \wedge \mathscr{T}^k \wedge \mathscr{C}_{(k+1)+} \arrow[rr, "\simeq"'] & & \mathscr{C}_{(k+1)+}.
\end{tikzcd}$$

Thus, we have a homotopy commutative diagram of $\mathscr{S}$-operads

$$\begin{tikzcd}
\mathscr{L} \arrow[d, "\epsilon^k"'] & & & \\
\mathscr{J}^k \wedge \mathscr{L} \arrow[r] \arrow[d, "\epsilon"'] & \mathscr{J}^k \wedge \mathscr{T}^k \wedge \mathscr{C}_{(k+1)+} \arrow[r, "\simeq"] & \mathscr{C}_{(k+1)+} \arrow[d] & \\
\mathscr{J}^{k+1} \wedge \mathscr{L} \arrow[r] & \mathscr{J}^{k+1} \wedge \mathscr{T}^{k+1} \wedge \mathscr{C}_{(k+2)+} \arrow[r, "\simeq"] & \mathscr{C}_{(k+2)+} \arrow[r] & \mathscr{C}_+
\end{tikzcd}$$

and hence taking the colimit,

$$\mathscr{L} \to \mathscr{C}_+$$

factors as

$$\begin{tikzcd}
\mathscr{L} \arrow[d, "\varinjlim \epsilon^k"'] \arrow[dr] & \\
(\varinjlim \mathscr{J}^k) \wedge \mathscr{L} \arrow[r] & \mathscr{C}_+ .
\end{tikzcd}$$

However, the lower left corner operad is equivalent to $\mathscr{E}$. ◻

Proposition 9.

$$\mathscr{L}(n) \simeq \bigvee_{(n-1)!} S^0.$$

Proof. We have a sequence of fibrations

$$\begin{tikzcd}
& \vdots \\
\bigvee_2 S^{k-1} \arrow[r] & \mathscr{C}_k(3) \arrow[d] \\
\bigvee_1 S^{k-1} \arrow[r] & \mathscr{C}_k(2) \arrow[d] \\
& \mathscr{C}_k(1) \simeq *.
\end{tikzcd}$$

The general fibration is

$$\bigvee_{n-1} S^{k-1} \to \mathscr{C}_k(n) \to \mathscr{C}_k(n-1).$$

Considering the corresponding Gysin cofibration

$$\mathscr{C}_k(n)_+ \to \mathscr{C}_k(n-1)_+ \to \mathscr{T}_k(n),$$

we see that $\mathscr{T}_k(n)$ has a based CW-decomposition with cells corresponding to $(n-1)$ copies of the cells of the CW-decomposition of $\mathscr{C}_k(n-1)_+$, suspended by k. Thus, stably, $\Sigma^{1-k}\mathscr{C}_k(n)_+$ has a finite CW-decomposition with $(n-1)!$ cells in dimension 0, and other cells in dimension $\leq 1-k$.

Now investigate the homotopy limit of the sequence
(23)

$$\ldots \to \Sigma^{(1-(k+1))(n-1)}\mathscr{C}_{k+1}(n)_+ \to \Sigma^{(1-k)(n-1)}\mathscr{C}_k(n)_+ \to \ldots \to \mathscr{C}_1(n)_+.$$

We see by obstruction theory that for each k, there exists a $K_k >> 0$ such that we have a stable factorization

$$\begin{tikzcd} \bigvee_{(n-1)!} S^0 \arrow[r, "Id"] \arrow[dr, dotted] & \bigvee_{(n-1)!} S^0 \\ \Sigma^{1-K_k}\mathscr{C}_{K_k}(n)_+ \arrow[r] \arrow[u, dotted] & \mathscr{C}_k(n)_+. \arrow[u] \end{tikzcd}$$

This implies both that the top cells of $\mathscr{C}_k(n)_+$ stably split (which is well known) and also that the homotopy limit (23) factors through

$$\ldots \xrightarrow{Id} \bigvee_{(n-1)!} S^0 \xrightarrow{Id} \bigvee_{(n-1)!} S^0$$

which implies the statement. $\square$

Proposition 10. *There is an equivalence between the* 1*-operad* $\mathscr{L}$ *and* $\mathscr{J} \wedge Ch$ *where* Ch *is the Ching operad* [6].

Proof. (sketch) One knows by imitating the proof of [10] that the Koszul dual, in the sense of Ching [6], of $\mathscr{C}_{k+}$ is $\mathscr{T}^{k-1} \wedge \mathscr{C}_k$. While Koszul duality, in general, does not commute with homotopy inverse limits, it follows by direct calculation that it does so in this case of $\mathscr{L}$. Thus, the Koszul dual of $\mathscr{L}$ is equivalent to $\mathscr{T}$, as is the Koszul dual of $\mathscr{J} \wedge Ch$. Applying Koszul duality again gives the statement. $\square$

3.2. **Derived Lie algebra representations.** By a *Lie algebra over S*, we shall mean an algebra g over the $\mathscr{S}$-operad $\mathscr{L}$. By a *g-representation*, we shall mean a module over the operad $\mathscr{L}$ and a Lie algebra g. Over an ordinary (tired old) ring, it is almost trivial to see that a representation of a Lie algebra g is the same thing as a left (or, alternately, right) module over its universal enveloping algebra Ug.

Over S, this is still true, but it requires more discussion. The universal enveloping algebra functor is the pushforward

$$U = u_\sharp$$

where

$$u : \mathscr{L} \to \mathscr{C}_{1+}$$

is the canonical map (see (22)). However, an operad module over $\mathscr{C}_{1+}$ models a *bimodule*, not a left or right module. Nevertheless, for a $\mathscr{C}_{1+}$-algebra A, a left (or right) $(\mathscr{C}_{1+}, A)$-module can be defined. Recall that connected components define an equivalence of operads in spaces:

$$\mathscr{C}_1 \to \Sigma$$

where Σ is the associative operad,

$$\Sigma(n) = \Sigma_n.$$

Denote by $\mathscr{C}_1^L(n)$ the fiber of $\mathscr{C}_1(n)$ over the isotropy subgroup Σ_n^L of n in Σ_n (isomorphic to Σ_{n-1}). Then a *left $(\mathscr{C}_{1+}, A)$-module* has structure maps

$$\mathscr{C}_1^L(n)_+ \wedge A^{n-1} \wedge X \to X$$

which satisfy associativity and equivariance with respect to Σ_n^L. Note that the operad structure map takes

$$\mathscr{C}_1^L(n) \times \mathscr{C}_1(k_1) \times \cdots \times \mathscr{C}_1(k_{n-1}) \times \mathscr{C}_1^L(k_n) \to \mathscr{C}_1^L(k_1 + \cdots + k_n).$$

The definition of a *right $(\mathscr{C}_{1+}, A)$-module* is analogous when we replace "n" by "1" and write R instead of L.

Proposition 11. *Let $\widetilde{\mathscr{C}}_{1+}$ be a cofibrant model of $\mathscr{C}_{1+}$. (We already assume $\mathscr{L}$ to be cofibrant). Let g be a cofibrant Lie algebra. Then the derived category of g-representations is canonically equivalent to the derived category of left (or, alternately, right) Ug-modules.*

Proof. Consider first the monad

$$D(g, X) = (Lg, D_1(g, X))$$

in the category of pairs (g, X) of spectra (by which we mean a product of two copies of the category of spectra) defining "$\mathscr{L}$-algebra g and $(\mathscr{L}, g)$-module X". Now consider the monad

$$E(A, X) = (\widetilde{C}_{1+}A, E_1(A, X))$$

in pairs of spectra (A, X) defining "$\widetilde{\mathscr{C}}_{1+}$-algebra A and left $(\widetilde{\mathscr{C}}_{1+}, A)$-module X". Then we have

$$D_1(g, X) = \bigvee_{n \geq 1} \mathscr{L}(n) \wedge_{\Sigma_{n-1}} g^{\wedge(n-1)} \wedge X$$

and

$$E_1(A, X) = \bigvee_{n \geq 1} \widetilde{\mathscr{C}}^L_{1+}(n) \wedge_{\Sigma_{n-1}} A^{\wedge(n-1)} \wedge X.$$

One checks from the definition of the map $\mathscr{C}_{(k+1)+} \to \mathscr{J} \wedge \mathscr{C}_{k+}$ that the map $\mathscr{L} \to \mathscr{C}_{1+}$ actually induces a map

$$D_1(g, X) \xrightarrow{\sim} E_1(g, X) \tag{24}$$

which is an equivalence. (This is plausible because of Proposition 9; however, note that more is being claimed here, namely certain diagram formed using the explicitly defined maps commutes on the nose.) From this, we obtain a map of monads

$$D \to E \tag{25}$$

which is an equivalence on the second coordinate. Now the two functors between the categories of g-representations and left Ug-modules are pushforward, and pullback along (25), composed with the unit map

$$g \to u^* Ug.$$

When (g, X) is a free D-algebra, (24) implies that the map on the second coordinate induced by (25) is an equivalence. For a general cofibrant D-algebra (g, X), we can use the two-sided bar construction $B(D, D, (g, X))$ and the associated filtration spectral sequence in the standard way.

The proof for right modules is analogous. □

The example given in the following proposition is fundamental for the development of our theory. By an *abelian* Lie algebra we shall mean an S-Lie algebra equivalent to an S-Lie algebra obtained by pullback from the map of $\mathscr{S}$-operads

$$\mathscr{L} \to \mathscr{E}.$$

(Perhaps, a more precise term than "abelian" would be "E_∞", but in this context it is overloaded.)

Proposition 12. *Let g be a cofibrant abelian Lie algebra and let R be an E_∞-algebra (i.e. a $\mathscr{C}_+$-algebra). Then a map of spectra*

$$\lambda : g \to R$$

canonically determines a g-representation with underlying spectrum R.

When R is equivalent to S as an E_∞-algebra, we shall refer to such a representation as a *character*, and denote it by S^λ. For a general E_∞-algebra R, we will speak more generally of R-*characters*, and use the notation R^λ.

Proof of Proposition 12: Consider the diagram (21). As usual, we obtain a morphism

$$u_\sharp p^* \to q^* i_\sharp \tag{26}$$

from its adjoint

$$p^* \to u^* q^* i_\sharp = p^* i^* i_\sharp,$$

which is p^* composed with the unit of the adjunction $(i_\sharp, i^*)$. Now (26) for an abelian cofibrant S-Lie algebra g can be written as a map of $\mathscr{C}_{1+}$-algebras

$$Ug \to q^* C_+ g \tag{27}$$

where C_+ is the monad associated to $\mathscr{C}_+$. Now an operad algebra is always an operad module over the same algebra. Thus, a map of spectra $g \to R$ determines by adjunction a map of $\mathscr{C}_+$-algebras

$$C_+ g \to R,$$

and hence a $(\mathscr{C}_+, C_+ g)$-module structure on R. By adjunction, this then determines a $(\mathscr{C}_{1+}, q^* C_+ g)$-module structure on R, hence, by (27), a (C_{1+}, Ug)-module structure and hence a left (or alternately right) Ug-module structure. Now use Proposition 11. $\square$

Proposition 13. *(The projection formula) Returning to the diagram (21), on a cofibrant $\mathscr{E}$-algebra X, the canonical morphism*

$$u_\sharp p^* X \to q^* i_\sharp X$$

(alternately, thinking of X as an abelian Lie algebra, $UX \to C_+ X$), is an equivalence.

Proof. By a colimit argument, it suffices to consider the case when X is a finite cell spectrum. By modeling the map of interest as

$$B(C_{1+}, L, X) \to C_+ X, \tag{28}$$

we see that (28) is, in the category of spectra, a wedge sum of maps of bounded below spectra, so we may use homology. In homology with coefficients in $\mathbb{Q}$, (28) induces an isomorphism just by the ordinary PBW theorem. With coefficients in $\mathbb{Z}/p$, the homology of LY is E_∞-Quillen homology of the abelian commutative $\mathbb{Z}/p$-algebra $H_*(Y, \mathbb{Z}/p)$. This was calculated in [30, 33]. The answer is the free Lie algebra on the Koszul dual to the Dyer-Lashof algebra. Essentially, they can be thought of as Steenrod operations without the admissibility relations. Using the Grothendieck spectral sequence as in [31], we see that (28) induces an isomorphism in mod p homology. □

Proposition 14. *Suppose X, $X_1, \dots, X_m$ are $\mathscr{L}$-algebras which are finite as spectra, and*

$$\phi_i : X_i \to X$$

are morphisms of $\mathscr{L}$-algebras such that the map of spectra

$$\bigvee_{i=1}^{m} \phi_i : \bigvee_{i=1}^{m} X_i \to X$$

is an equivalence of spectra. Then the map of spectra

$$\bigwedge_{i=1}^{m} UX_i \to UX$$

given by $U\phi_i$ and $\mathscr{C}_{1+}$-multiplication is an equivalence.

Proof. This proposition is proved by the same method as the previous one, using a calculation of homology. □

3.3. **Products.** The operad tensor product of modules over a cofibrant operad $\mathscr{C}$ in spectra is defined as follows: Let C be the associated monad. Then define

$$C_{(1)}(R, M) = \bigvee_{n \geq 0} \mathscr{C}(n+1) \wedge_{\Sigma_n} R^{\wedge n} \wedge M,$$

$$C_{(2)}(R, M, N) = \bigvee_{n \geq 0} \mathscr{C}(n+2) \wedge_{\Sigma_n} R^{\wedge n} \wedge M \wedge N.$$

Then

$$\underline{C}_{(1)}(R, M) = (CR, C_{(1)}(R, M))$$

is a monad in pairs of spectra, defining "$\mathscr{C}$-algebra A, $(\mathscr{C}, A)$-module".

$$\underline{C}_{(1,1)}(R, M, N) = (CR, C_{(1)}(R, M), C_{(1)}(R, N))$$

is a monad in triples of spectra, defining "$\mathscr{C}$-algebra A, $(\mathscr{C}, A)$-modules M, N". The functor

$$\underline{C}_{(2)}(R, M, N) = (CR, C_{(2)}(R, M, N))$$

is now a (left $\underline{C}_{(1)}$, right $\underline{C}_{(1,1)}$)-functor from triples to pairs of spectra. The *operad tensor of* $(\mathscr{C}, R)$*-modules* M*,* N is defined as

$$M \otimes_{(\mathscr{C},R)} N = B(C_{(2)}, \underline{C}_{(1,1)}, (R, M, N)).$$

It is a $(\mathscr{C}, B(C, C, R))$-module, so without further work, this operation does not have good point-set properties, but it works in the derived category.

In the derived category, the operad tensor product is commutative, but the question of associativity and unitality is tricky. In general, neither holds, although we may readily define

$$\left(\bigotimes_{i=1}^{n}\right)_{\mathscr{C},R} M_i$$

for any $n \geq 0$ in the same way, and we have canonical comparison maps in the derived category

$$\left(\bigotimes_{j=1}^{k}\right)_{\mathscr{C},R} \left(\left(\bigotimes_{i=1}^{n_j}\right)_{\mathscr{C},R} M_{j,i}\right) \to \left(\bigotimes_{i,j}\right)_{\mathscr{C},R} M_{i,j}. \tag{29}$$

For $\mathscr{C} = \mathscr{C}_{\infty+}$, the comparison maps (29) are equivalences (the first version of [9] was written using this fact), and it follows from Koszul duality that the operad tensor product over $\mathscr{L}$ is associative and unital also. On the other hand, it is easy to observe that the operad tensor product of an empty set of $(\mathscr{C}, R)$-modules is always equivalent to R, which, as we will see, is *not* the unit for $\otimes_{\mathscr{L},g}$ (the "trivial representation" is).

Lemma 15. *For representations* M*,* N *of a cofibrant* S*-Lie algebra* g*,*

$$M \otimes_{(\mathscr{L},g)} N \sim M \wedge N$$

as spectra.

Proof. One checks that the natural composition map

$$\mathscr{L}(2) \wedge L_{(1)}(R, M) \wedge L_{(1)}(R, M) \to L_{(2)}(R, M, N) \tag{30}$$

is an equivalence (again, this requires Proposition 9 and some care in matching terms). The key observation is that

$$L_{(1)}(R, M) \simeq \bigvee_{k \geq 1} M,$$

whereas

$$L_{(2)}(R,M,N) = \bigvee_{n\geq 2}(\bigvee_{n-1} M\wedge N) = (\bigvee_{k\geq 1} M)\wedge(\bigvee_{\ell\geq 1} N)$$

(as $\Sigma_{n-1}/\Sigma_{n-2}$ is a set of $n-1$ elements).

This establishes the statement of the Lemma in the case of a free module over a free Lie algebra. Now (30) is, by definition, a right $L_{(1,1)}$-functor, so we can use the usual bar construction argument. $\square$

From now on, we shall denote the product "$\otimes_{(\mathscr{L},g)}$" also simply as "$\wedge$".

Theorem 16. *(Products of characters) Let g be a cofibrant abelian S-Lie algebra, let R_1, R_2 be cofibrant E_∞-algebras, and let*

$$\phi_i : g \to R_i,\ i = 1,2$$

be maps of spectra. Define

$$\psi = \phi_1 \wedge 1 + 1\wedge \phi_2 : g \to R_1 \wedge R_2. \tag{31}$$

Then we have a canonical equivalence in the derived category

$$(R_1)^{\phi_1} \otimes_{(L,g)} (R_2)^{\phi_2} \sim (R_1\wedge R_2)^{\psi}. \tag{32}$$

Note that in our category, a smash product of E_∞-algebras is canonically an E_∞-algebra.

We will need a number of steps to prove the theorem.

Lemma 17. *Let R be a cofibrant $\mathscr{C}_{\infty+}$-algebra. Then the derived category of left (alternatively, right) q^*R-modules is canonically equivalent to the derived category of $(\mathscr{C}_{\infty+}, R)$-modules.*

Proof. We have a map induced by q:

$$C^L_{1+(1)}(R,M) \to C_{\infty+(1)}(R,M).$$

This gives a functor from $(\mathscr{C}_{\infty+}, R)$-modules to $(\mathscr{C}^L_{1+}, R)$-modules. Next, we note that free modules are equivalent in both categories, since for $\mathscr{C}_{\infty+}$-algebras R, the canonical map

$$B(C_{1+(1)}(?,M), C_{1+}, R) \to B(C_{\infty+(1)}(?,M), C_{\infty+}, R) \tag{33}$$

is an equivalence. Denoting the target as $\mathfrak{G}$, then

$$M \mapsto B(\mathfrak{G}, C_{1+(1)}(B(C,C,R),?), M)$$

is the desired inverse functor on the level of derived categories. $\square$

Next, we introduce a $\otimes$-product of left (C_{1+}, R_i)-modules, $i = 1, 2$, which coincides with the *external* smash-product in the case $\mathscr{C}_{\infty+}$-algebras R_i. Define, in effect,

$$C^L_{1+(1)}(R, M) = \bigvee_n \mathscr{C}^L_{1+}(n+1) \wedge_{\Sigma_n} R^n \wedge M$$

where $\mathscr{C}^L_1(n+1)$ is the fiber of $\mathscr{C}_1(n+1)$ over the isotropy group of $n+1$ in Σ_{n+1}. Define also

$$C^L_{1+(2)}(R_1, R_2, M, N) = \bigvee_{k,\ell \geq 0} \mathscr{C}^{LL}_{1+}(k, \ell) \wedge_{\Sigma_k \times \Sigma_\ell} R_1^{\wedge k} \wedge R_2^{\wedge \ell} \wedge M \wedge N$$

where $\mathscr{C}^{LL}_1(k, \ell) \subseteq \mathscr{C}_1(k+\ell+2)$ is the fiber over the subgroup of $\Sigma_{k+\ell+2}$ of permutations preserving the blocks $\{1, \dots, k\}$, $\{k+1, \dots, k+\ell\}$ and fixing $k+\ell+1$, $k+\ell+2$. Then we have a monad on quadruples of spectra

$$\underline{C}^L_{1+(1,1)}(R_1, R_2, M_1, M_2) = (C_{1+}R_1, C_{1+}R_2, C^L_{1+}(R_1, M_1), C^L_{1+}(R_2, M_2))$$

defining "$\mathscr{C}_{1+}$-algebra R_i and left R_i-module M_i, $i = 1, 2$" and, of course, again, a monad in pairs

$$\underline{C}^L_{1+(1)}(R, M) = (C_{1+}R, C^L_{1+(1)}(R, M))$$

defininig "$\mathscr{C}_{1+}$-algebra R and left R-module M".

We also have a functor

$$C^{\otimes}_{1+}(R_1, R_2) = \bigvee_{k,\ell \geq 0} \mathscr{C}^{\otimes}_{1+}(k, \ell) \wedge_{\Sigma_k \times \Sigma_\ell} R_1^k \wedge R_2^\ell$$

where $\mathscr{C}^{\otimes}_1(k, \ell)$ is the fiber of $\mathscr{C}_1(k+\ell)$ over the subgroup of $\Sigma_{k+\ell}$ preserving the blocks $\{1, \dots, k\}$, $\{k+1, \dots, k+\ell\}$. Then $C^{\otimes}_{1+}$ is a left C_{1+}, right (C_{1+}, C_{1+})-functor and

$$(R_1, R_2) \mapsto B(C^{\otimes}_{1+}, (C_{1+}, C_{1+}), (R_1, R_2))$$

is a model for the external smash product of C_{1+}-algebras. We will denote it by $R_1 \wedge_1 R_2$ (the underlying operation on spectra is, indeed, "$\wedge$").

Now defining

$$\underline{C}^L_{1+(2)} = (C^{\otimes}_{1+}, C^L_{1+(2)}),$$

this is a (left $\underline{C}^L_{1+(1)}$, right $\underline{C}^L_{1+(1,1)}$)-functor, so

$$B(C^L_{1+(2)}, \underline{C}^L_{1+(1,1)}, (R_1, R_2, M_1, M_2))$$

is the exterior smash-product $M_1 \wedge_1 M_2$, as a left $R_1 \wedge R_2$-module.

Defining, more simply,

$$C^{\otimes}_{\infty+(2)}(R_1, R_2, M_1, M_2) = \bigvee_{k,\ell \geq 0} \mathscr{C}_{\infty+}(k+\ell+2) \wedge_{\Sigma_k \times \Sigma_\ell} R_1^{\wedge k} \wedge R_2^{\wedge \ell} \wedge M \wedge N,$$

$$C^{\otimes}_{\infty+}(R_1, R_2) = \bigvee_{k,\ell \geq 0} \mathscr{C}_{\infty+}(k+\ell) \wedge_{\Sigma_k \times \Sigma_\ell} R_1^k \wedge R_2^\ell,$$

we can analogously define external smash product of $\mathscr{C}_{\infty+}$-algebras

$$(R_1, R_2) \mapsto B(C^{\otimes}_{\infty+}, (C_{\infty+}, C_{\infty+}), (R_1, R_2))$$

which we will, for the moment, denote by $\wedge_\infty$. Then there is a morphism of $\mathscr{C}_{1+}$-algebras

$$R_1 \wedge_1 R_2 \to R_1 \wedge_\infty R_2$$

which is an equivalence (we omit q^* on the left hand side). Similarly, for modules, define

$$\begin{aligned}&\underline{C}_{\infty+(1,1)}(R_1, R_2, M_1, M_2) = \\ &(C_{\infty+}R_1, C_{\infty+}R_2, C_{\infty+(1)}(R_1, M_1), C_{\infty+(1)}(R_2, M_2))\end{aligned}$$

and

$$M_1 \wedge_\infty M_2 = B(C^{\otimes}_{\infty+(2)}, \underline{C}_{\infty+(1,1)}, (R_1, R_2, M_1, M_2))$$

is a smash product functor from $\mathscr{C}_{\infty+}R_i$-modules M_i, $i = 1, 2$, to $\mathscr{C}_{\infty+}$-$R_1 \wedge_\infty R_2$-modules. Again, for $\mathscr{C}_{\infty+}$-algebras R_i, there is a canonical map of $\mathscr{C}_{1+}$-modules from the $\mathscr{C}_{1+}$-smash product of left R_i-modules to the smash product of $\mathscr{C}_{\infty+}$-modules, which is an equivalence.

Rejoining Lie algebras. A proof of Theorem 16: Now analogously to the comparison of Lie algebra representations and left (alternately, right) modules over the universal enveloping algebra, we obtain a natural map

$$Lg \to C^{\otimes}_{1+}(g, g).$$

From this, we get a map

$$B(L, L, g) \to B(C^{\otimes}_{1+}(g, g), L, g),$$

which can be interpreted as a map of S-Lie algebras

$$g \to Ug \wedge_1 Ug. \tag{34}$$

Using the adjunction, this gives a map of $\mathscr{C}_{1+}$-algebras (a "non-rigid Hopf algebra structure")

$$Ug \to Ug \wedge_1 Ug. \tag{35}$$

Also for modules, we get maps

$$L_{(1)} \to C^L_{1+(1)},$$

$$L_{(2)} \to C^L_{1+(2)}$$

and comparing the 2-sided bar constructions of monads, one shows that the pullback of a $\wedge_1$-product of left Ug-modules M_1, M_2 via (35) is equivalent to $M_1 \otimes_{(L,g)} M_2$.

Now when g is abelian, (34) expands to

$$g \longrightarrow Ug \wedge_1 Ug \xrightarrow{\sim} Cg \wedge Cg \xrightarrow{\sim} C(g \vee g). \tag{36}$$

(The last equivalence can again be seen directly using our methods, or alternately it follows from commutation of left adjoints with coproducts.) By inspection, the composition (36) is homotopic, as a map of spectra, to the composition

$$g \xrightarrow{Id \times Id} g \times g \xleftarrow{\sim} g \vee g \xrightarrow{\eta} C(g \vee g).$$

Now (36), by adjunction, gives

$$Cg \to C(g \vee g).$$

The composition

$$Ug \xrightarrow{\sim} Cg \longrightarrow C(g \vee g),$$

in the derived category of $\mathscr{C}_{1+}$-algebras, is homotopic to (35) by uniqueness of adjoints (U is a Quillen left adjoint). We have proved that the diagram

$$\begin{tikzcd} Ug \arrow[r] \arrow[d] & Ug \wedge_1 Ug \arrow[d] \\ Cg \arrow[r] & Cg \wedge_\infty Cg \end{tikzcd}$$

is commutative up to homotopy in $\mathscr{C}_{1+}$-algebras. Composing with a smash product of $\mathscr{C}_{\infty+}$-morphisms

$$Cg \to R_i,\ i = 1, 2,$$

which are classified by morphisms of spectra

$$g \to R_i$$

completes the proof of the Theorem. □

Corollary 18. *Let g be an abelian S-Lie algebra and let $\lambda : g \to S$ be a character. Then*

$$S^\lambda \wedge S^{-\lambda} \sim S^0$$

in the derived category of g-representations. In particular, S^λ is invertible and hence strongly dualizable in this category with respect to this symmetric monoidal structure.

□

4. The S-Lie algebra gl_nS and its representations. The restricted category.

4.1. **Verma modules.** To construct $\mathscr{C}_{1+}$-algebras in $\mathscr{S}$, it suffices by [8] to construct multifunctors from the "associative" operad Σ to the multi-category *Perm* of permutative categories. By the results of [11], it suffices to construct a *weak* multifunctor, i.e. up to coherence isomorphisms with appropriate coherence diagram (see also [21]). (Note: Technically, the target of the Elmendorf-Mandel functor is symmetric spectra, but a version of it also exists which lands in the category used in the present paper - see [28].)

The $\mathscr{C}_{1+}$-algebra gl_nfSet where $fSet$ denotes the category of finite sets and bijections, sends the unique object to the product of $n \times n$ copies of the category $fSet$. Then the morphism functor

$$gl_nfSet \times \cdots \times gl_nfSet \to gl_nfSet$$

is given by "multiplication of matrices" where by "multiplication" we mean Cartesian product and by "addition" we mean disjoint union.

In fact, this construction can be generalized: Let

$$T \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$$

be a set of pairs which is transitive as a binary relation. Then

$$g_TfSet = \prod_{(i,j)\in T} fSet$$

is a weak multifunctor from the associative operad Σ to the multicategory *Perm* and thus applying the Elmendorf-Mandell functor $\mathscr{K}$, we obtain a $\mathscr{C}_{1+}$-algebra, hence an $\mathscr{L}$-algebra, g_TS. There are a number of examples of $\mathscr{L}$-algebras g_TS which will be useful to us, for example the Borel subalgebra b_+S and its opposite b_-S of upper (resp. lower) triangular matrices, and the corresponding nilpotent "Lie subalgebras" n_+S and n_-S, and similarly for any parabolic subalgebras, and also the Cartan algebra of diagonal matrices h_nS.

Now considering the $\mathscr{C}_{1+}$-subalgebra b_+fSet which consists of upper triangular $n \times n$ matrices in $fSet$ (embedded into gl_nfSet by sending the remaining entries into the empty set). In turn, b_+fSet projects onto the algebra h_nfSet of *diagonal* $n \times n$ matrices in $fSet$ (forgetting the above-diagonal terms). Applying the Elmendorf-Mandell realization functor $\mathscr{K}$ [8, 11, 21], we obtain a diagram of $\mathscr{C}_{1+}$-algebras:

$$\begin{tikzcd} b_+S \arrow[r, "\alpha"] \arrow[d, "\pi"'] & gl_nS \\ h_nS & \end{tikzcd} \tag{37}$$

We further note that the $\mathscr{C}_{1+}$-algebra structure on h_nS is the pull-back of a $\mathscr{C}_{\infty+}$-structure, as the Σ-structure on h_nfSet comes from the Čech resolution $E\Sigma$ by the results of Elmendorf-Mandell [8] (note that h_nfSet is the product of n copies of $fSet$ where the operations are done component-wise).

Now pull the diagram (37) back to Lie algebras via u^*. By what we just observed, h_nS is an abelian Lie algebra, and hence by Proposition 12 above, an n-tuple of integers $\lambda = (k_1, \dots, k_n)$ (specifying homotopy classes of maps $S \to S$) specifies a representation S^λ of h_nS. Let $\widetilde{\pi^*S^\lambda}$ be a cofibrant replacement of π^*S^λ. Put

$$V_\lambda = \alpha_\sharp\widetilde{\pi^*S^\lambda}.$$

This is the *Verma module* over gl_nS induced from the character λ. (Whenever considering this on the nose, we will automatically assume cofibrant replacement has been performed, without indicating it in the notation.)

In accordance with conventions of representation theory [32], however, when using numerical subscripts, we perform a *ρ-shift*: We put

$$(\overline{k}_1, \dots, \overline{k}_n) = (k_1, \dots, k_n) + \rho,$$

where

$$\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2})$$

is half the sum of all positive roots. When indexing by numbers, we then write

$$V_{(\overline{k}_1, \dots, \overline{k}_n)} = V_\lambda.$$

It is necessary to point out, however, that for n even, ρ is not an integral weight. Therefore, stritly speaking, Verma modules $V_{(\overline{k}_1, \dots, \overline{k}_n)}$ for n even will exist only up to shifting the weight by $(a, \dots, a)$ where $a \in (1/2) + \mathbb{Z}$.

4.2. **Variants of the construction. The graded category and the p-complete category.** We shall also refer to the derived category of gl_nS-representations as defined so far as the *unrestricted* category of representations. While we will need to use this category, and will prove some results about it, at present a complete calculation of even a single non-zero *Ext*-group in this category is out of reach. Typically, such a group is a homotopy group of the dual DX of an infinite spectrum X, where we, perhaps, have some control over the homology of X.

Because of that, we shall also consider some variants of the derived category of representations which are more treatable. One tool we shall often use is *completion at p*. By this, we mean Bousfield localization of $\mathscr{S}$ at the Moore spectrum $M\mathbb{Z}/p$. As shown in [9], localization of this type commutes with the forgetful functor from algebras over a cofibrant operad in $\mathscr{S}$ to $\mathscr{S}$, and hence all of our constructions, at least on the derived level, can be readily carried over to the p-complete category $\mathscr{S}_p$. We will typically specialize our calculations to the p-complete category at $p >> k$, which means that p is larger than some constant multiple of k.

The other important variant of the category of gl_nS-representations is obtained as follows. The entire construction of the $\mathscr{C}_{1+}$-algebra (and therefore $\mathscr{L}$-algebra) gl_nS, and the Verma module, can be made Λ-graded, where Λ is the weight lattice. Here gl_n is graded by the roots, i.e. diagonal entries have degree 0, and $e_{i,j}$ has degree

$$(0,\dots,0,1,0,\dots,0,-1,0,\dots,0)$$

where the 1 is in the i'th and -1 is in the j'th position. In this notation, the weight lattice is

$$\Lambda = \mathbb{Z}^n. \tag{38}$$

Matrix multiplication is additive with respect to this grading. This occurs over $fSet$ just the same as over an ordinary ring. Therefore, we can apply a $\mathbb{Z}^n$-graded version of the Elmendorf-Mandell functor (agreeing, in fact, to sum only homogeneous terms in the same degree), creating a $\mathbb{Z}^n$-graded version of a $\mathscr{C}_{1+}$-algebra, and $\mathscr{L}$-algebra. Obviously, all of the definitions (such as representations) and results established so far can be carried to the Λ-graded context.

4.3. **Function objects.** Function objects are generally obtained as right adjoints to versions of the smash product. Several different flavors of such functors arise in our context. For an S-Lie algebra g and a g-representation V, we denote by $\mathscr{F}_g(V,?)$ the right adjoint to $V\wedge ?$, where $\wedge$ is the internal smash product. We will be typically interested in the case when V is cofibrant, in which case this functor coincides with its right derived functor. On the other hand, if X is a spectrum, and V is a g-representation, then the ordinary smash product $X\wedge V$ clearly has a canonical structure of a g-representation. We denote the right adjoint to this functor by $F_g(V, ?)$ from g-representations to spectra. Again, in the case of V cofibrant, this is the same as the corresponding right derived functor. Two comments are in order. First, note that the functor F_g can be generalized to modules over general operad algebras, and also to left (resp. right) modules over a $\mathscr{C}_{1+}$-algebra A. In this last case, we denote the function object by F_A and observe that the right derived funtors of F_g and F_{Ug} coincide. The other comment is that the functors F_g and $\mathscr{F}_g$ are in fact related. The point is that there is a trivial representation functor from spectra to g-representations, which has a right adjoint, which we may think of as "g-invariants". On the level of right derived functors, then, F_g is the g-invariants of $\mathscr{F}_g$.

We will denote the graded versions of these functors by the superscript 0, i.e. we write $\mathscr{F}_g^0$, F_g^0, etc.

4.4. **Morphisms of Verma modules. Blocks.** We begin with morphisms of characters. From now on, we shall sometimes write h for $h_n S$. We will also sometimes omit the S in the notation for S-Lie algebras.

Theorem 19. *Let $\lambda, \mu : h \to S$ be morphisms in $\mathscr{S}$ (determining characters by Proposition 12). Then*

$$F_h^0(S^\lambda, S^\mu) = \begin{matrix} 0 & \text{if } \lambda \neq \mu \\ DC_{\infty+}(\Sigma h) & \text{if } \lambda = \mu \end{matrix} \tag{39}$$

where Σ denotes the suspension of a spectrum and the characters are considered to be concentrated in the degree given by their weight.

Proof. Since h is in degree 0, $\lambda \neq \mu$ implies

$$F^0(Uh^{\wedge k} \wedge S^\lambda, S^\mu) = 0.$$

where F^0 denotes the function object (a spectrum) in the category of graded spectra. Consequently,

$$F_{Uh}^0(B(Uh, Uh, S^\lambda), S^\mu) = 0.$$

Consider therefore the case $\lambda = \mu$. By the graded version of Corollary 18, it suffices to consider the case $\lambda = \mu = 0$. In this case, we can use the adjunction

$$F_h^0(S, S) = F_{C_{\infty+}h}(S, S) = F_{C_{\infty+}h}(S, F(S, S)) = F(S \wedge_{C_{\infty+}h} S, S).$$

Now

$$S \wedge_{C_{\infty+}h} S \sim C_{\infty+}\Sigma S,$$

as claimed. ◻

Comment: The dual $DC_{\infty+}(\Sigma h) = F(C_{\infty+}\Sigma h, S)$ can be calculated. It is the product of spectra of the form

$$DE\Sigma_{k+} \wedge_{\Sigma_k} S^{k\alpha} \tag{40}$$

where α is the sign representation of Σ_k. (Recall that EG for a finite group G is a free G-CW complex which is non-equivariantly contractible.) Now using Spanier-Whitehead duality, (40) can be rewritten as

$$F(E\Sigma_{k+}, S^{-k\alpha})^{\Sigma_k}, \tag{41}$$

which, by Carlsson's theorem [5], is the completion at the augmentation ideal of the Burnside ring of the fixed point spectrum

$$(S^{-k\alpha})^{\Sigma_k}. \tag{42}$$

Now we have a cofibration sequence

$$(\Sigma_k/A_k)_+ \to S^0 \to S^\alpha \tag{43}$$

where A_k is the alternating group, and its Spanier-Whitehead dual

$$S^{-\alpha} \to S^0 \to (\Sigma_k/A_k)_+. \tag{44}$$

This means that $S^{-k\alpha}$ is the iterated fiber of the cube

$$\bigwedge_k (S^0 \to (\Sigma_k/A_k)_+). \tag{45}$$

Since taking fixed points preserves iterated homotopy fibers, $(S^{-k\alpha})^{\Sigma_k}$ is obtained by applying Σ_k-fixed points to the corners of the cube (45), which are S^0, or wedges of copies of $(\Sigma_k/A_k)_+$, and their fixed point spectra are

$$\bigvee_{G\subset\Sigma_k} BG_+$$

and wedges of copies of

$$\bigvee_{G\subset A_k} BG_+.$$

Theorem 20. *Suppose a_i, b_i, $i = 1,\dots,n$ are two sequences of nonnegative integers. Then, in the p-complete graded category,*

$$F^0_{gl_n}(V_{(a_1,\dots,a_n)}, V_{(b_1,\dots,b_n)}) \sim *$$

unless

$$(a_1,\dots,a_n) \geq (b_1,\dots,b_n) \tag{46}$$

(where $\geq$ denotes the ordering of weights, which is just the componentwise ordering) and

$$\begin{array}{l}\textit{There exists a permutation } \sigma \textit{ on } \{1,\dots,n\} \textit{ such}\\ \textit{that } (a_1,\dots,a_n) \equiv (b_{\sigma(1)},\dots,b_{\sigma(n)}) \mod p.\end{array} \tag{47}$$

Comment: We can show an analogous result with F^0 replaced by F and condition (46) omitted, but it is substantially more difficult, and there seems to be no benefit for our purposes in this paper.

Proof. First, by standard "change of rings", we have

$$F^0_{gl_n}(V_{\lambda_1}, V_{\lambda_2}) \sim F^0_{b_+}(S^{\lambda_1}, V_{\lambda_2}). \tag{48}$$

Now using further the fact that S^{λ_1} is a pullback of an h-representation, the right hand side of (48) can be further written as

$$F^0_h(S^{\lambda_1}, F^*_{b_+}(Uh, V_{\lambda_2})). \tag{49}$$

By the "$*$" superscript, we mean the graded object whose terms are F^0's where the source is smashed with the character of the negative of the desired weight. This functor is the right adjoint to the forgetful functor from graded b_+-representations to graded h-representations. Now again

$$F^*_{b_+}(Uh, V_{\lambda_2}) \sim F^*_{n_+}(S^{\lambda_1}, V_{\lambda_2}). \tag{50}$$

This is the graded "nilpotent stable homotopy" of V_{λ_2}. On the right hand side of (50), the significance of the "λ_1"-decoration of S is only for grading.

Now all the weights of n_+ are positive. This means that in each given degree, the cosimplicial object calculating the given term is, in fact, finite (in the sense that it has only finitely many non-trivial cosimplicial degrees, and each degree is a finite spectrum).

This means that we can calculate with homology. Working in the p-complete category, specifically for $H = H\mathbb{Z}/p$, we have

$$HF_{n_+}(S, V_\lambda) \sim F_{Hn_+}(H, HV_\lambda). \tag{51}$$

Calculating the right hand side is pure (E_∞-) algebra. It is, in fact, almost the same as doing the calculation in ordinary algebra in characteristic p [20], with the exception that we have to include a discussion of higher Dyer-Lashof operations. However, Dyer-Lashof operations in mod p homology occur in weights which are multiples of p, and hence can be ignored for our purposes.

Let us calculate, then, in (tired old) characteristic p. We will proceed by induction on n. Consider the abelian graded Lie subalgebra of n_+ corresponding to the binary relation

$$\{(1, j) \mid j = 2, \dots, n\}.$$

Then we have a short exact sequence of Lie algebras

$$0 \to a \to n_+ \to n'_+ \to 0$$

where n'_+ is the graded Lie algebra corresponding to the transitive relation

$$\{(i,j) \mid 2 \leq i < j \leq n\}.$$

Then we have a Hochschild-Serre spectral sequence

$$H^p(n'_+, H^q(a, V_\lambda)) \Rightarrow H^{p+q}(n_+, V_\lambda).$$

To consider the action of a on $V_\lambda \sim Un_-$, we note that $e_{1,i}$ acts non-trivially on $e_{i,1}$. Let a_- be the graded Lie algebra corresponding to the transitive relation

$$\{(i,1) \mid 1 < i \leq n\}.$$

Considering elements of Un_- of the form

$$e_{2,1}^{k_2} \dots e_{n,1}^{k_n},$$

and considering the actions of $e_{2,1}, \dots, e_{n,1}$ in order (i.e., using a sequence of $n-1$ consecutive Hochschild-Serre spectral sequences), we see that cocycles are only in the requisite weights by reducing to the $n = 2$ case. The $n = 2$ case is an easy direct calculation.

Thus, we have

$$H^*(a, V_\lambda) \cong H^*(a, Ua_-) \otimes Un'_- \tag{52}$$

where n'_- is defined symmetrically to n'_+. To compute the cohomology of n'_+ acting on (52), the weight shifting action of n'_+ on $H^*(a, Ua_-)$ can be neglected by a filtration spectral sequence. The statement then follows from the induction hypothesis. □

Therefore, at least for Verma modules of regular weights

$$V_{(a_1,\dots,a_n)},\ a_i \geq 0,\ i \neq j \Rightarrow a_i \neq a_j,$$

in the p-complete category with $p >> a_i$, there are no non-zero morphisms in the derived category between Verma modules in different blocks (i.e. $V_{(a_1,\dots,a_n)}$ and $V_{(b_1,\dots,b_n)}$ where the sequence $(a_1, \dots, a_n)$ is not a permutation of the sequence $(b_1, \dots, b_n)$.

4.5. **Some special finite gl_k-representations.** There is a natural (weak) action of $gl_k fSet$ on $(fSet)^k$. Rectifying and applying the $\mathscr{K}$-functor of Elmendorf and Mandell [8], we obtain the "standard representation" $W = W_k$ of $gl_k S$. This representation is graded. It is useful to note that for any graded representation U, if $0 \leq m < p$ and we are working in the p-complete category, we can construct the m'th *symmetric product*

$$Sym^m U = (E\Sigma_{m+}) \wedge_{\Sigma_m} U^{\wedge m} \tag{53}$$

and the *exterior product*

$$\Lambda^m U = \Sigma^{-m} Sym^m(\Sigma U). \tag{54}$$

Note that the restriction $m < p$ is not necessary, but in this range the mod p homology of $B\Sigma_m$ is $\mathbb{Z}/p$ in dimension 0, and hence (53), (54) are finite (and of the expected dimension).

In particular, for $k < p$, we have a $gl_k S$-representation

$$Det = \Lambda^k(W_k).$$

We will often use smash powers of the Det representation. As already remarked, the underlying graded spectrum of $Det_k = Det$ is S in the appropriate degree. "Non-integral smash powers of Det" can also be constructed by the following trick: A representation on S can be considered as a morphism of S-Lie algebras of the given S-Lie algebra into the $\mathscr{C}_{1+}$-algebra $F(\widetilde{S}, \widetilde{S})$. But the identity inclusion

$$S \to F(\widetilde{S}, \widetilde{S}) \tag{55}$$

is an equivalence, and hence $F(\widetilde{S}, \widetilde{S})$ is, in the derived category, canonically q^* of the E_∞-algebra S. Now for any m prime to p, we can compose (55) with the character

$$m : C_{\infty+}\widetilde{S} \to S$$

do create the "m'th determinant power representation" which we will denote by $Det^{\wedge m}$. Placed into the appropriate degree, it is a special graded representation.

4.6. **Some useful pairs of adjoint functors.** Consider a transitive relations $T \subseteq T'$ on $\{1, \dots, n\}$. Then we have an "inclusion" morphism of graded Lie algebras

$$\kappa_{g_T, g_{T'}} = \kappa_{T,T'} : g_T \to g_{T'} \tag{56}$$

Of course, we have the pullback functor $\kappa^*_{T,T'}$ from graded $g_{T'}$-representations to graded g_T-representations. This functor has a left adjoint $(\kappa_{T,T'})_\sharp$ and a right adjoint $(\kappa_{T,T'})_*$ (left and right Kan extension). Certain compositions of these functors will be of major use to us. Specifically, we will often consider the case when

$$g_T = gl_{k_1} \times \cdots \times gl_{k_m},\ g = g_{T'} = gl_n \tag{57}$$

with $k_1 + \cdots + k_m = n$. Let p_+ resp. p_- be the parabolic Lie subalgebra in gl_n with reductive part g_T with respect to the positive (resp. negative) roots.

4.7. **co-Verma modules.** It turns out that to capture the analogue of derived Zuckermann functors, we will actually have to go even further and investigate algebraic groups over S. On representations of algebraic groups, however, one naturally has *induction*, which is an analogue of the functors κ_*. The functors $\kappa_\sharp$ do not in general have analogues. Because of this, rather than Verma modules, it will be easier for us to work with generalized *co-Verma modules*, which are modules of the form

$$(\kappa_{p_+,g})_* \pi^* W$$

where π is the projection from the parabolic to its Levi factor, and W is a representation of the Levi factor.

Note: in classical representation theory, it is customary to swap p_+ for p_- so that the Verma and co-Verma modules are in the same category. We do not bother with this convention here. A part of the reason is that even in the graded sense, the graded pieces of our Verma modules are not truly finite spectra (since they will be extended products), and therefore dualization is not as nicely behaved as one may hope. When we work in the category completed at a large prime p, however, in weights whose ρ-shifted coordinates are non-negative integers much smaller than p, graded morphisms behave well in the given range.

5. Commutative Hopf algebras and Harish-Chandra pairs over S

In this section, we will define the S-module version of Harish-Chandra pairs and their representations.

5.1. **Commutative Hopf algebras over S.** A *commutative Hopf algebra R over S* is a $\mathscr{C}_{1+}$-coalgebra in the category of commutative S-algebras, i.e. explicitly, a choice of an operad $\mathscr{A}$ in $\mathscr{B}$ equivalent to $\mathscr{C}_{1+}$, and structure maps in the category of commutative S-algebras

$$\epsilon : \mathscr{A}(0) \widetilde{\otimes} R \to S,$$

$$\psi : \mathscr{A}(n) \widetilde{\otimes} R \to \underbrace{R \wedge \cdots \wedge R}_{n \text{ times}}$$

where $\widetilde{\otimes}$ is the based Kelly tensor product [15] (which can be always taken between a based simplicial set and an object of a symmetric monoidal category with simplicial realization preserving the symmetric monoidal structure) which satisfy the usual operad coalgebra relations.

A *comodule algebra* (more generally, a $\mathscr{C}$-comodule algebra for a $\mathscr{B}$-operad $\mathscr{C}$) over a commutative S-Hopf algebra R is a commutative R-algebra (resp. $\mathscr{C}$-algebra) A together with a structure morphism

$$\theta : \mathscr{A}(n)^R \widetilde{\otimes} A \to A \wedge R \wedge \cdots \wedge R, \tag{58}$$

(where $\mathscr{A}(n)^R$ is as in Section 3.2), compatible with ψ and ϵ in the obvious sense. Obviously, R is always a comodule algebra over itself. We will also be interested in comodules in the category of spectra, which are spectra whose structure is defined by the same formula (58), where $\widetilde{\otimes}$ now denotes the Kelly product in the category of spectra (which is, essentially, the smash product). For a fixed commutative S-Hopf algebra R, denote by R-*Comod* the category of R-comodules.

The smash product creates a commutative associative unital product in the category of R-comodules and also in the category of R-comodule algebras by

$$\begin{array}{c}
\mathscr{A}(n)^R \widetilde{\otimes} V \wedge W \\
\Big\downarrow{\scriptstyle \Delta \wedge Id} \\
(\mathscr{A}(n)^R \wedge \mathscr{A}(n)^R) \widetilde{\otimes} V \wedge W \\
\Big\downarrow{\scriptstyle T} \\
(\mathscr{A}(n)^R \widetilde{\otimes} V) \wedge (\mathscr{A}(n)^R \widetilde{\otimes} W) \\
\Big\downarrow{\scriptstyle \theta \wedge \theta} \\
V \wedge R \wedge \cdots \wedge R \wedge W \wedge R \wedge \cdots \wedge R \\
\Big\downarrow{\scriptstyle T} \\
V \wedge W \wedge (R \wedge R) \wedge \cdots \wedge (R \wedge R) \\
\Big\downarrow{\scriptstyle Id \wedge \phi \wedge \cdots \wedge \phi} \\
V \wedge W \wedge R \wedge \cdots \wedge R
\end{array}$$

where T denotes switches of factors and ϕ is the product in the category of commutative S-algebras. Using the adjoint functor theorem, we find that in the category of R-comodules, the functor $? \wedge V$ has a right adjoint $\mathscr{F}_R(V, ?)$. However, it is important to note that this right adjoint may not have the expected properties, in particular the underlying spectrum may not be $F(V, ?)$. This is because we do not have conjugation as a part of our definition of commutative Hopf algebra (the reason of which, in turn, is that we do not know how to construct examples with rigid conjugation - we will return to this point below). In the absence of conjugation, we do not expect the function object to behave as expected. To give a very simple analogy, consider the category of complex representations of the commutative monoid $\mathbb{N}_0$ (think of it multiplicatively, writing the generator as t). Then the unit of the tensor product is the "trivial" representation $\mathbb{C}_1$ where t acts by 1. Now consider the representation $\mathbb{C}_0$ where t acts by 0. We actually have $\mathscr{F}(\mathbb{C}_0, \mathbb{C}_1) = 0$, because tensoring any representation with $\mathbb{C}_0$ makes t act by 0.

Note that $\mathscr{F}_R(V, V)$ is canonically a $\mathscr{C}_{1+}$-comodule algebra in the category of R-comodules, while $F(V, V)$ is a $\mathscr{C}_{1+}$-algebra in $\mathscr{S}$. If $\mathscr{U}$ is the forgetful functor from the category of R-comodules to $\mathscr{S}$, there further is a canonical morphism of $\mathscr{C}_{1+}$-algebra

$$\mathscr{U}\mathscr{F}_R(V, V) \to F(V, V), \tag{59}$$

but it is not an equivalence for general R.

By a *morphism of commutative S-Hopf algebras*, we shall mean a morphism of $\mathscr{C}_{1+}$-coalgebras in the $\mathscr{C}_{\infty+}$-algebras. Similarly, we define morphisms of comodules and comodule algebra by requiring compatibility with the structure morphisms θ.

Our typical example of a commutative Hopf algebra is, for a transitive relation T on $\{1, \dots, n\}$, a commutative S-Hopf algebra

$$\mathscr{O}_{G_T} = det^{-1} C_\infty(g_T^\vee). \tag{60}$$

Here we use the notation $h^\vee = F(h, S)$, we suppress cofibrant replacement from the notation, and det^{-1} means inverting the determinant 0-homotopy class. Both cofibrant replacement and inverting a homotopy class can be done in a way which does not spoil the $\mathscr{C}_{1+}$-coalgebra structure, using the methods of [9]. This structure is given by the diagram

$$\begin{tikzcd}
g_T^\vee \arrow[rr] \arrow[d, "\eta"'] & & g_T^\vee \wedge g_T^\vee \arrow[d, "\eta\wedge\eta"] \\
C_\infty(g_T^\vee) \arrow[r, dotted] & C_\infty(g_T^\vee) \wedge C_\infty(g_T^\vee), &
\end{tikzcd}$$

which follows from adjunction. From the structure,

$$V = \bigvee_n S$$

is a $\mathscr{O}_{GL_n S}$-comodule, which we call the *standard representation.* Also, S is canonically a comodule, which we will call the *trivial representation.* As before, the smash product of comodules can be used to construct other comodules. We shall be especially interested in the comodules

$$\Lambda^q V = \Sigma^{-q} E\Sigma_{q+} \wedge_{\Sigma_q} V^{\wedge q}. \tag{61}$$

In this paper, we will be interested in completing at a large prime p, by which we mean Bousfield localizing at the Moore spectrum $M\mathbb{Z}/p$ ([9]). We denote by $(\mathcal{O}_{GL_nS}\text{-}Comod)_p$ the full subcategory of $\mathcal{O}_{GL_nS}\text{-}Comod$ on $M\mathbb{Z}/p$-local comodules, and by

$$X \to X_p \tag{62}$$

the corresponding localization map of $(\mathcal{O}_{GL_nS}\text{-}Comod)_p$-comodules. One sees easily that the underlying morphism of spectra of (62) is also $M\mathbb{Z}/p$-localization. Recall that the category of $M\mathbb{Z}/p$-local spectra has a smash product obtained by applying the smash product and then $M\mathbb{Z}/p$-localizing. It is commutative, associative and unital up to homotopy, the unit being S_p. We do not know how to rigidify this product, however. There is also a right adjoint, which is simply $F(?,?)$ restricted to $M\mathbb{Z}/p$-local spectra. The category $(\mathcal{O}_{GL_nS}\text{-}Comod)_p$ has a smash product obtained by applying the ordinary smash product and then $M\mathbb{Z}/p$-localizing. We will denote both this smash product and the underlying smash product of $M\mathbb{Z}/p$-local spectra by $\wedge_p$. The unit is the $M\mathbb{Z}/p$-localized trivial representation.

We do not know how to construct Bousfield localization for commutative S-Hopf algebras. In fact, the reader should consult [9] to see that even for commutative algebras, Bousfield localization is not entirely what we expect. For example, one cannot p-complete the unit.

This is the reason why we need results such as the following

Lemma 21. *Let $p >> n$ be a prime. Then V_p is strongly dualizable in the category of $M\mathbb{Z}/p$-local $\mathcal{O}_{GL_nS}$-comodules, and we have equivalences of spectra*

$$\mathcal{U}\mathcal{F}_{\mathcal{O}_{GL_nS}}(V_p, S_p) \xrightarrow{\sim} F(V_p, S_p) \tag{63}$$

$$\mathcal{U}\mathcal{F}_{\mathcal{O}_{GL_nS}}(V_p, V_p) \xrightarrow{\sim} F(V_p, V_p). \tag{64}$$

Proof. We first prove that

$$(\Lambda^n S)_p \tag{65}$$

is invertible in the derived category of $(\mathcal{O}_{GL_nS}\text{-}Comod)_p$. In fact, it is true in general that if an ($M\mathbb{Z}/p$-local) R-comodule L forgets to S (resp. S_p) and specifies a (homotopy) invertible class in R (resp. R_p), then L is invertible in the derived category of R-comodules. To see this, let L^{-1} be the inverse of L in the category of ($M\mathbb{Z}/p$-local) spectra. We recall from [7] that an R-comodule structure on M can be specified by a "structure map"

$$M \to M \wedge R \tag{66}$$

and the vanishing of a series of obstructions

$$S^k \wedge M \to M \wedge \underbrace{R \wedge \cdots \wedge R}_{k+1 \text{ times}}. \tag{67}$$

In the case of L^{-1}, we specify (66) as the R-inverse homotopy class to the homotopy class associated with the comodule structure (66) for L. The homotopies making the obstructions (67) vanish are then computed from smashing (point-wise) with the corresponding homotopies for L, and noting that S (resp. S_p) is also a comodule using the unit of R.

We shall now prove (63); (64) is proved analogously. We have a morphism of $\mathcal{O}_{GL_nS}$-comodules

$$V \wedge \Lambda^{n-1}V \to \Lambda^n V,$$

and hence

$$V \wedge \Lambda^{n-1}V \to (\Lambda^n V)_p. \tag{68}$$

Thus, we have a morphism of $\mathcal{O}_{GL_nS}$-comodules

$$V_p \wedge_p (\Lambda^{n-1}V)_p \wedge_p ((\Lambda^n V)_p)^{-1} \to S_p. \tag{69}$$

Similarly, transfer gives a morphism of $\mathcal{O}_{GL_nS}$-comodules

$$\Lambda^n V \to (V \wedge \Lambda^{n-1}V)_p, \tag{70}$$

which gives a morphism of $\mathcal{O}_{GL_nS}$-comodules

$$S_p \to V_p \wedge_p (\Lambda^{n-1}V)_p \wedge_p ((\Lambda^n V)_p)^{-1}. \tag{71}$$

One further sees that (69) and (71) forget to the unit and counit of strong duality in $M\mathbb{Z}/p$-local spectra, and hence define a unit and counit of strong duality in the derived category of $(\mathcal{O}_{GL_nS}\text{-}Comod)_p$ up to equivalence. Strong duality and (63) follow. $\square$

Let us make a few more remarks on the theory of commutative S-Hopf algebras.

Lemma 22. *Let R be a commutative Hopf algebra over S. Then there is a canonical structure of a $(\mathscr{C}_{k+1})_+$-coalgebra on the k'th bar construction $B^k(R)$, and a canonical structure of a $\mathscr{L}$-coalgebra (where $\mathscr{L}$ is the Lie operad on QR, where Q denotes topological Quillen homology).*

Proof. First of all, recall that topological Quillen homology can be defined by Dwyer-Kan stabilization:

$$QR = \underset{\rightarrow}{\mathrm{holim}}\, \Sigma^{-k} B^k R$$

where by Σ^{-k} we mean the k'th desuspension on the augmentation ideal (right adjoint to the bar construction). Therefore, if we show that $B^k(R)$ has a canonical structure of a $(\mathscr{C}_{k+1})_+$-coalgebra, $\Sigma^{-k} B^k R$ has a canonical structure of a $\Sigma^{-k}(\mathscr{C}_{k+1})_+$-coalgebra, we know that QR has a structure of a $\mathscr{L}$-coalgebra by (19).

The proof that $B^k R$ has a canonical structure of a $(\mathscr{C}_{k+1})_+$-coalgebra is analogous to [12]. □

We see that if V is an R-comodule, then V is also a right comodule over the co-Lie algebra QV.

Lemma 23. *One has*

$$Q\mathscr{O}_{G_T} \sim g_T^\vee$$

as co-Lie algebras.

Proof. For a $\mathscr{C}_{1+}$-coalgebra X, one has

$$QC_\infty X \sim X$$

as an S-Lie coalgebra (recall the morphism of operads $\mathscr{L} \to \mathscr{C}_{1+}$). Inverting the determinant class does not affect the Quillen homology. (It disappears after one bar construction.) □

5.2. **Harish-Chandra pairs and their representations.** Harish-Chandra pairs appear in many areas of representation theory (for example algebraic, compact Lie, affine, super), see [19]. The basic idea is obvious - we want a notion of a group representation which is simultaneously, and compatibly, a representation of a Lie algebra. One must be mindful, however, of subtle details of the definition which change depending on the context.

In this paper, by a *pre-Harish-Chandra pair* (R, g), we mean simply a commutative S-Hopf algebra R, and an R-equivariant S-Lie algebra g. More generally, for an operad $\mathscr{C}$ in $\mathscr{S}$, and a commutative S-Hopf algebra R, an *R-equivariant $\mathscr{C}$-algebra* X is defined as an object of $\mathscr{S}$

which has both the structure of an R-comodule and a $\mathscr{C}$-algebra, such that the following diagram commutes for $n \geq 2$:

(72)
$$\begin{tikzcd}
\mathscr{A}(m+1)^R \wedge \mathscr{C}(n) \wedge X \wedge \cdots \wedge X \arrow[r] \arrow[d, "T\circ(\Delta\wedge Id)"] & \mathscr{A}(m+1)^R \wedge X \arrow[ddd] \\
\mathscr{C}(n) \wedge \mathscr{A}(m+1)^R \wedge X \wedge \cdots \wedge \mathscr{A}(m+1)^R \wedge X \arrow[d] & \\
\mathscr{C}(n) \wedge X \wedge \underbrace{R \wedge \cdots \wedge R}_{m} \wedge \cdots \wedge X \wedge \underbrace{R \wedge \cdots \wedge R}_{m} \arrow[d, "(Id\wedge\phi)"] & \\
\mathscr{C}(n) \wedge X \wedge \cdots \wedge X \wedge R \wedge \cdots \wedge R \arrow[r] & X \wedge \underbrace{R \wedge \cdots \wedge R}_{m}
\end{tikzcd}$$

Here for simplicity, we denote all shuffles by T, all diagonals by Δ and all products by ϕ.

A *morphism* f from an R_1-equivariant $\mathscr{C}$-algebra X_1 to an R_2-equivariant $\mathscr{C}$-algebra X_2 is defined as a morphism of commutative S-Hopf algebras
$$f_R : R_2 \to R_1$$
and a morphism of $\mathscr{C}$-algebras
$$f_X : X_1 \to X_2$$
which satisfy the obvious commutative diagram. In particular, this defines morphisms of pre-Harish-Chandra pairs. Notice that we put the contravariance into the commutative Hopf algebra coordinate. This is because we think of the Hopf algebras as coordinate rings of algebraic groups, in which morphisms would be ordinarily written contravariantly.

If R is a commutative Hopf algebra over S and $\mathscr{C}$ is an operad in $\mathscr{S}$, and X is an R-equivariant $\mathscr{C}$-algebra, then we can define an R-equivariant $(\mathscr{C}, X)$-module Y as an R-comodule which is also a $(\mathscr{C}, X)$-module, and the obvious analogue of diagram (72) where we replace in each entry, one X by Y, commutes. Morphisms are again defined in the obvious way.

For a pre-Harish-Chandra pair (R, g), an (R, g)- *representation* (or (R, g)-*module*) is an R-equivariant $(\mathcal{L}, g)$-module. The category of (R, g)-modules (with fixed commutative S-Hopf algebra R and R-equivariant Lie algebra g) will be denoted by (R, g)-Mod.

By a *Harish-Chandra pair* (R, g, γ), (when no confusion arises, we shall sometimes omit the γ from the notation), we mean a pre-Harish-Chandra pair (R, g) together with a morphism of S-Lie coalgebras

$$\gamma : R \to g^\vee. \tag{73}$$

A *morphism of Harish-Chandra pairs*

$$(R_1, g_1, \gamma_1) \to (R_2, g_2, \gamma_2)$$

is a morphism of pre-Harish Chandra pairs where we have a commutative diagram of S-Lie coalgebras

$$\begin{tikzcd} R_2 \arrow[r, "f_R"] \arrow[d, "\gamma"'] & R_1 \arrow[d, "\gamma"] \\ g_2^\vee \arrow[r, "f_g^\vee"] & g_1^\vee. \end{tikzcd}$$

A *representation* of (or *module* over) a Harish-Chandra pair (R, g, γ) is an R-equivariant $(\mathscr{L}, g)$-module whose underlying $(\mathscr{L}, R)$-comodule structure (coming from the R-equivariant structure) coincides with the $(\mathscr{L}, R)$-comodule structure coming from (73). Modules over a Harish-Chandra pair form a full subcategory of the category of modules over the underlying pre-Harish-Chandra pair.

For any commutative S-Hopf algebra R and any R-comodule W, $\mathscr{F}_R(W, W)$ is, by adjunction, a R-equivariant associative algebra (and hence R-equivariant S-Lie algebra). We will be especially interested in the case when $R = \mathscr{O}_{GL_n S}$, and $W = V$ is the standard representation. Then by Lemma 21, at least up to the eyes of $M\mathbb{Z}/p$, we can think of $\mathscr{F}_{\mathcal{O}_{GL_n S}}(V_p, V_p)$ as an $\mathscr{O}_{GL_n S}$-model of $gl_n S$, which we already know is its Lie algebra by Quillen cohomology (Lemma 23).

For our purposes, though, we will need to consider a somewhat more general example. Concretely, we will have

$$n = k_1 + \cdots + k_m, \tag{74}$$

(which we will refer to as an *ordered partition* $\mathbf{k} = (k_1, \ldots, k_m)$ *of* n) and

$$R = R_{\mathbf{k}} = \mathscr{O}_{GL_{k_1} \times \cdots \times GL_{k_m} S}.$$

We will also have some

$$i_1 + \cdots + i_s = m,$$

and

$$l_j = k_{i_0 + \cdots + i_{j-1} + 1} + \cdots + k_{i_1 + \cdots + i_j}. \tag{75}$$

(We will refer to the ordered partition $\mathbf{l} = (l_1, \dots, l_s)$ of (75) as a *refinement* of the ordered partition (74).) Let W be the standard GL_nS-representation. Then as an R-comodule, W has an increasing filtration Φ where $\Phi^j W$ is the standard $GL_{\ell_1+\dots+\ell_j}S$-representation. We may define a category of filtered R-comodules in the obvious way, and obtain the R-equivariant associative algebra (hence, S-Lie algebra)

$$\mathbf{p} = \mathbf{p_l} = \mathscr{F}_R^{\Phi}(W_p, W_p) \tag{76}$$

where the superscript means the function object in the filtered R-comodule category. The notation $\mathbf{p}$ stands for *parabolic*, as these are examples of parabolic Lie subalgebras of gl_nS.

Lemma 24. *With the above notation, we have an equivalence*

$$\mathscr{F}_R^{\Phi}(W_p, W_p) \to \mathscr{F}^{\Phi}(W_p, W_p) \tag{77}$$

where the right hand side denotes the analogous construction in the category of spectra.

Proof. One sees that, more or less by definition, in the category of R-comodules,

$$\mathscr{F}_R^{\Phi}(W_p, W_p) \sim \bigvee_{j_1 \le j_2} \mathscr{F}_R((W_{k_{j_1}})_p, (W_{k_{j_2}})_p)$$

where W_{k_j} are the pushforwards of the basic comodules over $\mathcal{O}_{GL_{k_j}S}$ to R. $\square$

In fact, all the morphisms of Harish-Chandra pairs we will consider will be of the form

$$\kappa : (R_{\mathbf{k_1}}, \mathbf{p_{l_1}}) \to (R_{\mathbf{k_2}}, \mathbf{p_{l_2}}) \tag{78}$$

where $\mathbf{k_1}$ is a refinement of $\mathbf{k_2}$ and $\mathbf{l_1}$ is a refinement of $\mathbf{l_2}$.

A variant of this construction is if we replace the filtration Φ by *grading*, which means taking

$$\ell = \ell_{\mathbf{l}} = \mathscr{F}_R(V, V)$$

where V is the standard representation of $R_{\mathbf{l}}$ (product of the standard representations of the $GL_{\ell_j}S$'s). We will also consider the morphism

$$\pi : (R_{\mathbf{k_1}}, \mathbf{p_l}) \to (R_{\mathbf{k_1}}, \ell_{\mathbf{l}}). \tag{79}$$

We refer to $\ell_{\mathbf{l}}$ as the *Levi factor* of the corresponding parabolic $\mathbf{p_l}$.

There are forgetful functors from

$$U_{R,g} : (R, g)\text{-}Mod \to R\text{-}Comod,$$
$$U_R : R\text{-}Comod \to \mathscr{S},$$

$$V_{R,g} : (R,g)\text{-}Mod \to g\text{-}Mod,$$
$$V_g : g\text{-}Mod \to \mathscr{S}.$$

All of these functors have right adjoints, and both are comonadic. The comonad $\mathscr{H} = \mathscr{H}_{(R,g)}$ corresponding to the composition

$$U = U_R \circ U_{R,g} = V_g \circ V_{R,g}$$

is called the *Hecke comonad* corresponding to the Harish-Chandra pair (R,g).

For a morphism of Harish-Chandra pairs

$$f : (R_1, g_1) \to (R_2, g_2) \tag{80}$$

we have a "forgetful functor"

$$f^* : (R_2, g_2)\text{-}Mod \to (R_1, g_1)\text{-}Mod. \tag{81}$$

Proposition 25. *The functor (81) preserves equivalences, and there is a functor*

$$f_* : (R_1, g_1)\text{-}Mod \to (R_2, g_2)\text{-}Mod \tag{82}$$

which is its right adjoint on the level of derived categories. Additionally, if $R_1 = R_2$, we have a commutative diagram up to equivalence:

$$\begin{tikzcd}
(R, g_1)\text{-}Mod \arrow[r, "f_*"] \arrow[d, "V_{(R,g_1)}"] & (R, g_2)\text{-}Mod \arrow[d, "V_{(R,g_2)}"] \\
g_1\text{-}Mod \arrow[r, "\phi_*"] & g_2\text{-}Mod
\end{tikzcd} \tag{83}$$

where $\phi : g_1 \to g_2$ is the underlying morphism of S-Lie algebras.

Proof. The functor f_* is constructed by the 2-sided cobar construction of Hecke comonads

$$f_*(?) = Cobar(?, \mathscr{H}_{(R_1,g_1)}, \mathscr{H}_{(R_2,g_2)}).$$

The commutativity of diagram (83) is essentially due to the fact that U_R takes the smash of R-comodules to the smash product of spectra. $\square$

Suppose we are given an $(R_{\mathbf{k}}, \ell)$-module W where ℓ is the Levi factor of a parabolic $\mathbf{p_l}$. (Note that the standard representation of ℓ, and extended powers of its suspensions, are examples of such representations W.) Suppose the ordered partition $\mathbf{l}$ is a refinement of another ordered partition $\mathbf{m}$. Then

$$V_{\mathbf{l},\mathbf{m},W} = \kappa_* \pi^* W \tag{84}$$

where κ is the morphism of Harish-Chandra pairs (78) with $\mathbf{l_1} = \mathbf{l}$, $\mathbf{l_2} = \mathbf{m}$, $\mathbf{k_1} = \mathbf{k_2} = \mathbf{k}$, and π is the morphism of Harish-Chandra pairs (79) where $\mathbf{k_1} = \mathbf{k}$. We refer to the representation (84) of the Harish-Chandra pair $(R_{\mathbf{k}}, \mathbf{p_m})$ as a *generalized co-Verma module.*

Example: Let us consider the case $n = 2$, $R_1 = \mathscr{O}_H$, $R_2 = \mathscr{O}_{GL_2}$, $\mathbf{k} = (1,1)$, $\mathbf{l} = (2)$. Consider the morphism

$$\kappa : (R_1, \mathbf{p_l}) \to (R_2, \mathbf{p_l}).$$

Let further on $(R_1, \mathbf{p_l})$, V^λ be the co-Verma module on a $\mathscr{O}_H$-character S^λ. We wish to compute

$$\kappa_*(V^\lambda). \tag{85}$$

As a warm-up, let us consider the classical case, i.e. let us work over $\mathbb{C}$. We can, of course, interpret this as a case of our setup, replacing $\mathbb{C}$ by the commutative S-algebra $H\mathbb{C}$. However, let us first work completely classically, i.e. with commutative $\mathbb{C}$-algebras in the tired old sense. Then the morphism of Hecke comonads is interpreted as a map of commutative graded Hopf algebras

$$\mathscr{O}_{GL_2} \to \mathscr{O}_B \otimes (Un_-)^\vee. \tag{86}$$

(As elsewhere in the paper, we set $B = B^+$, the subgroup of lower triangular matrices.)

Thus, we must study the Bruhat decomposition

$$B \times N_- \to GL_2. \tag{87}$$

On matrices, this is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b_{11} & xb_{11} \\ b_{21} & xb_{21} + b_{22} \end{pmatrix}.$$

Thus, the $\mathscr{O}_?$ of (87) can be written as

$$\begin{array}{rcl} a_{11} & \mapsto & b_{11} \\ a_{12} & \mapsto & xb_{11} \\ a_{21} & \mapsto & b_{21} \\ a_{22} & \mapsto & xb_{21} + b_{22}. \end{array} \tag{88}$$

Now in (86), Un_- should be thought of as a divided power algebra on generators

$$\gamma_n = \frac{x^n}{n!} \tag{89}$$

(although, of course, over $\mathbb{C}$, this is just $\mathbb{C}[x]$), so we can write (86) as

(90)

$$det^{-1}\mathbb{C}[a_{11}, a_{12}, a_{21}, a_{22}] \to (b_{11}b_{22})^{-1}\mathbb{C}[b_{11}, b_{21}, b_{22}] \otimes \mathbb{C}\{\gamma_0, \gamma_1, \gamma_2, \dots\}$$

given by (88). We see in any case from (88) that in the sub-$\mathbb{C}$-module of the target generated by monomials not containing b_{21} (which corresponds to co-Verma modules), the possible powers of x occurring in a monomial which has b_{11}^{ℓ} are

$$x^0, \dots, x^{\ell}.$$

This corresponds to the fact that $\kappa_*(V^{\lambda})$ is the irreducible GL_2-representation of weight λ.

To interpret this example over S, we note that basically one can argue the same way. One difficulty is that

$$Un_-^{\vee} = F(C_{\infty}S, S)$$

which has been calculated by Carlsson's theorem [5], but is in general more complicated than over $\mathbb{C}$. The other difficulty is the denominator in (89), which will cause disruptions at large weights.

Because of this, in the present paper, we restrict to a 'large prime' setting. Of course, the 'small prime' case is in principle more interesting, and we will return to it in future work. In the present example, we see that if we work in the p-completed category of spectra (localized at $M\mathbb{Z}/p$) and that the ρ-shifted components of the weight λ of our co-Verma module are non-negative integers $<< p$, then the argument goes through and we see that $\kappa_*(V^{\lambda})$ is a finite GL_2-representation of dimension $\ell + 1$ where the difference between the first and second coordinate of λ is ℓ.

5.3. **Localization.** When dealing with Harish-Chandra pairs of the form $(R_{\mathbf{k}}, \mathbf{p_l})$ and morphisms of the form (78) (resp. (79)), we have a variant of all the constructions described so far in this section where everything is graded by gl_n-weights. From now to the end of the present paper, we will work in this graded context. Additionally, we will work in the categories of $(R_{\mathbf{k}}, \mathbf{p_l})$-modules which have a highest weight (generalized Verma modules of the form (84) are an example, provided that W has a highest weight). Additionally still, we will Bousfield-localize the full subcategory of $(R_{\mathbf{k}}, \mathbf{p_l})$-modules with highest weight at $M\mathbb{Z}/p$ and we will always assume $k_i << p$. We will denote this category by

$$(R_{\mathbf{k}}, \mathbf{p_l})\text{-}hwMod.$$

But sometimes we wish to restrict attention even further, to full subcategories of objects which arise by taking fibrations and homotopy limits of co-Verma modules coming from characters S^{λ}, where λ is bounded in an appropriate sense.

This can be addressed through the concept of *localization*. Suppose we have a Harish-Chandra pair $(R_{\mathbf{k}}, \mathbf{p}_{\mathbf{l}})$, and a set

$$\mathfrak{E} \subseteq Obj((R_{\mathbf{k}}, \mathbf{p}_{\mathbf{l}})\text{-}hwMod).$$

We say that an object X of $(R_{\mathbf{k}}, \mathbf{p}_{\mathbf{l}})$-$hwMod$ is $\mathfrak{E}$-*local* if for every object Q of $(R_{\mathbf{k}}, \mathbf{p}_{\mathbf{l}})$-$hwMod$ which satisfies

$$(R_{\mathbf{k}}, \mathbf{p}_{\mathbf{l}})\text{-}hwMod_*(Q, E) = 0 \text{ for all } E \in \mathfrak{E},$$

we have

$$(R_{\mathbf{k}}, \mathbf{p}_{\mathbf{l}})\text{-}hwMod_*(Q, X) = 0.$$

(Here $?_*$ denotes the graded morphism group, i.e. allowing arbitrary $\mathbb{Z}$-suspensions of the objects involved.) The full subcategory of $(R_{\mathbf{k}}, \mathbf{p}_{\mathbf{l}})$-$hwMod$ on $\mathfrak{E}$-local objects will be denoted by $\mathfrak{E}$-$(R_{\mathbf{k}}, \mathbf{p}_{\mathbf{l}})$-$hwMod$.

Proposition 26. *Under the assumptions of the beginning of this subsection, the inclusion*

$$\mathfrak{E}\text{-}(R_{\mathbf{k}}, \mathbf{p}_{\mathbf{l}})\text{-}hwMod \subseteq (R_{\mathbf{k}}, \mathbf{p}_{\mathbf{l}})\text{-}hwMod$$

has a left adjoint $L_{\mathfrak{E}}$, called localization*. Additionally, given a morphism of Harish-Chandra pairs*

$$f : (R_1, g_1) \to (R_2, g_2)$$

satisfying the same assumptions as those of Proposition 25, let $E \subseteq (R_1, g_1)$-$hwMod$ and let

$$f_*(\mathfrak{E}) = \{f_*(E) \mid E \in \mathfrak{E}\}.$$

Then f_* restricts to a functor

$$f_{*,\mathfrak{E}} : \mathfrak{E}\text{-}(R_1, g_1)\text{-}hwMod \to f_*\mathfrak{E}\text{-}(R_2, g_2)\text{-}hwMod,$$

which is right adjoint to the functor

$$f^{*,\mathfrak{E}} : f_*\mathfrak{E}\text{-}(R_2, g_2)\text{-}hwMod \to \mathfrak{E}\text{-}(R_1, g_1)\text{-}hwMod$$

given by

$$f^{*,\mathfrak{E}}(X) = L_{\mathfrak{E}} f^*(X).$$

Proof. Formal from the definitions. □

Lemma 27. *If $X \in \mathfrak{E}$ then X is $\mathfrak{E}$-local, i.e. $L_{\mathfrak{E}}(X) \cong X$.*

Proof. Formal. □

In all the cases we will be interested in this paper from now on, we will work under the assumptions in the beginning of this subsection. We will consider the Borel subgroup B of GL_n of lower triangular matrices, with $\mathscr{O}_B$-equivariant parabolic Lie algebra $\mathbf{b}$ (the model constructed in (76)). We shall work in the $M\mathbb{Z}/p$-localized category of Harish-Chandra pair representations.

At this point, a comment must be made on the ρ-shift. It was noted in Subsection 4.1 that for gl_n, the components of the ρ-shift may not be integral, so we may need to add a half-integral multiple of the determinant weight to obtain integral components. It will be advantageous for us to do this once and for all. Let, therefore,

$$\rho' = (n-1, n-2, \dots, 0).$$

We shall, from now on, shift by ρ' instead of ρ.

The set $\mathfrak{E}_0 \subset (\mathscr{O}_H, \mathbf{b})\text{-}hwMod$ is the set of all modules of ρ'-shifted weights whose coordinates are in $\{0, \dots, k-1\}$ where $k << p$. All the classes of objects of $(R, g)\text{-}hwMod$ we will localize at will then be of the form $\mathfrak{E} = f_*(\mathfrak{E}_0)$ where $\mathfrak{E}_0$ is as defined above, and $f : (\mathscr{O}_H, \mathbf{b}) \to (R, g)$ is (the coefficients of) an inclusion of Harish-Chandra pairs.

In fact, using the same notation, we shall be interested the specific case where $R = \mathscr{O}_H$, $g = gl_n$. We then define the category $\mathfrak{O}_{n,k}$ (where k is as in the last paragraph) as the category of $(\mathscr{O}_H, gl_n)$-modules which can be filtered so the associated graded terms are $\mathbb{Z}$-suspensions of f_*S^λ with $S^\lambda \in \mathfrak{E}_0$ (i.e. co-Verma-modules). We also denote $\mathfrak{E} = f_*(\mathfrak{E}_0)$.

More generally, we will consider the case where $R = \mathscr{O}_L$ where L corresponds to a Levi factor of $\mathbf{p} \supseteq \mathbf{b}$ (always interpreted in the sense of (76)), which is a parabolic Lie subalgebra of gl_n, and $g = gl_n$. In this case, we define the category $\mathfrak{O}_{\mathbf{p},n,k}$ as the category of $(\mathscr{O}_L, gl_n)$-modules which can be filtered so the associated graded terms are $\mathbb{Z}$-suspensions of f_*S^λ with $S^\lambda \in \mathfrak{E}_0$ (i.e. generalized co-Verma-modules). In this case, we denote $\mathfrak{E} = \mathfrak{E}_{\mathbf{p}} = f_*(\mathfrak{E}_0)$.

Theorem 28. *Under the standing assumptions, let*

$$f = \mathscr{O}_\kappa : \mathscr{O}_{L_1} \to \mathscr{O}_{L_0}$$

where $\kappa : \mathbf{p}_0 \to \mathbf{p}_1$ is an inclusion of parabolics, and ℓ_1, ℓ_0 are the corresponding Levi factors. Then we have a functor

$$\kappa^* = L_{\mathfrak{E}_{\mathbf{p}_0}} Res_f : \mathfrak{O}_{\mathbf{p}_1,n,k} \to \mathfrak{O}_{\mathbf{p}_0,n,k} \tag{91}$$

whose left derived functor is left adjoint to a right derived functor of

$$\kappa_* = Ind_f : \mathfrak{O}_{\mathbf{p}_0,n,k} \to \mathfrak{O}_{\mathbf{p}_1,n,k}. \tag{92}$$

Additionally, the left derived functor of (91) also has a left adjoint denoted by

$$\kappa_\sharp, \tag{93}$$

and we have, on the level of derived functors,

$$\kappa_\sharp = \kappa_*[2(dim(\mathbf{p}_1) - dim(\mathbf{p}_0))] \tag{94}$$

where the functors (92) and (93) are referred to as right (resp. left) Zuckermann functor*.*

Proof. To prove the adjunction between (91) and (92), we first show that the functors are well-defined. In case of (92), we may as well think of f as a morphism of Harish-Chandra pairs

$$(\mathscr{O}_{L_0}, \mathbf{p}_1) \to (\mathscr{O}_{L_1}, \mathbf{p}_1) \tag{95}$$

(since the objects both in the source and target of (92) are obtained from this context by applying κ_* from $\mathbf{p}_1$ to gl_n. Now computing the κ_* of (95) on a co-Verma module is an extension of the Example at the end of last section: In the range of weights specified, we obtain a $\mathscr{O}_{L_1}$-comodule which is a wedge of m spheres where m is finite and equal to the dimension of the Levi factor representation we obtain when applying an analogous construction over $\mathbb{C}$. (Over S, we shall also refer to this as a *finite representation.*) This proves that (92) is well defined.

To treat (91), once again, we may work with the morphism of Harish-Chandra pairs (95) instead, but this time, we shall also consider the morphism

$$g : (\mathscr{O}_{L_0}, \mathbf{p}_0) \to (\mathscr{O}_{L_0}, \mathbf{p}_1). \tag{96}$$

To resolve $W' = f^*(W)$ where W is a finite representation in terms of co-Verma modules, consider first

$$g_* g^*(W'). \tag{97}$$

By the projection formula, (97) is equivalent to the $(\mathscr{O}_{L_0}, \mathbf{p}_1)$-module

$$g_* g^*(S) \wedge W' \tag{98}$$

where S is the trivial comodule. But now letting Cg denote the cone on an S-Lie algebra g in the category of S-Lie algebras, we may also consider the morphism of Harish-Chandra pairs

$$h : (\mathscr{O}_{L_0}, C\mathbf{p}_0) \to (\mathscr{O}_{L_0}, C\mathbf{p}_1), \tag{99}$$

and in particular we have a canonical morphism of $(\mathscr{O}_{L_0}, \mathbf{p}_1)$-modules from (98) to

$$h_* h^*(S) \wedge W'. \tag{100}$$

Moreover, the composite morphism

$$W' \to h_* h^*(S) \wedge W'$$

is an equivalence (the *dual Koszul resolution*). Filtering by the dimensions in the suspended cone coordinate, we see that the dual Koszul resolution has a decreasing filtration by co-Verma modules and moreover there is a morphism

$$h_* h^*(S) \wedge W' \to Q \tag{101}$$

where Q consists of finitely many filtered pieces in the specified weight range, and the homotopy fiber F of (101) is concentrated in lower weights. (F comes from the fact that $C_\infty S^{-1}$, completed at p, has an "exterior algebra" part similar as over $\mathbb{C}$, and then additional "derived" parts coming from extended p'th powers.)

In any case, this discussion implies that (101) is localization at $\mathfrak{E}$ by Lemma (27).

To prove the adjunction between (93) and (91), and (94), we use formal duality arguments together with calculations which go through because of the finiteness and large prime hypotheses.

Specifically, to manufacture a left adjoint out of a right adjoint, we need an invertible object ω and a morphism

$$\tau : \kappa_* \omega \to S. \tag{102}$$

In our case, we have

$$\omega = S[2(dim(\mathbf{p}_1) - dim(\mathbf{p}_0)] \tag{103}$$

and the appropriate morphism (102) is constructed from the rational case using the large prime hypothesis. Next, one proves the projection formula

$$\varrho : X \wedge \kappa_* \omega \xrightarrow{\sim} \kappa_*(\kappa^* X \wedge \omega). \tag{104}$$

One then sets

$$\kappa_\sharp(X) = \kappa_*(X \wedge \omega), \tag{105}$$

and the counit of adjunction is constructed as

$$\kappa_\sharp \kappa^* X \xrightarrow{Id} \kappa_*(\kappa^* X \wedge \omega) \xrightarrow{\rho^{-1}} X \wedge \kappa_* \omega \xrightarrow{\tau} X. \tag{106}$$

Validity of the triangle identities up to isomorphism is a calculation, again, mimicking the rational case using the large prime hypothesis.

One thing to note is that in our present setting, the objects ω and S, while they are comodules, are not objects of the categories $\mathfrak{O}_{\mathbf{p},n,k}$.

However, smashing with them is an endofunctor on $\mathfrak{O}_{\mathbf{p},n,k}$, which is precisely what we need. □

6. Khovanov cube and diagram relations

At this point, we start considering a smooth projection of an oriented link. We assume (just as [32]) that the crossings are at most double, that they are transverse and that the plane is given a system of Cartesian coordinates where in the neighborhood of each crossing, the two crossing strands are in the first and third resp. second and fourth quadrant when the origin is shifted to the point of the crossing. Furthermore, we assume that the strands are oriented upward (i.e. from negative to positive in the direction of the y-coordinate). We also assume that when the projection is tangent to any horizontal line (i.e. line parallel to the x axis), then the critical point is non-degenerate (i.e. it is a graph of a function with non-zero second derivative).

6.1. **The basic setup.** As in Sussan [32], we will assign to each oriented link projection as above a stable homotopy type (more precisely a finite spectrum). More generally, we will assign mathematical entities to oriented tangles projected into

$$\mathbb{R} \times [a_0, a_1],\ a_0 < a_1$$

with crossings as above where the only points with y-coordinate a_0 or a_1 are the ends, and the ends meet the horizontal lines transversely. Let

$$n_{a_i} = \sum n(s)$$

where the sum is over the strands ending on the line $y = a_i$, and $n(s) = 1$ resp. $n(s) = k - 1$ for an upward resp. downward oriented strand. Consider the graded S-Lie algebras $g_i = gl_{n_{a_i}}$, $i = 0, 1$. Consider the category

$$\mathfrak{O}_{n,k}$$

introduced after Lemma 27. Now consider the parabolic $\mathbf{p}_i \subseteq gl_{n_{a_i}}$ whose Levi factor consists of block sums of matrices with 1×1 block for every upward strand, and a $(k-1) \times (k-1)$ block for every downward strand. Using the machinery of Subsection 5.3, we will consider the parabolic BGG categories

$$\mathfrak{O}_{\mathbf{p}_i,i},\ i = 0, 1$$

with respect to the parabolic $\mathbf{p}_i$ inside the Lie algebra $gl_{n_{a_i}}$. (Throughout, we are working completed at a large prime.)

The "Khovanov cube" associated with a tangle projection as described will be an $\mathscr{S}$-enriched functor

$$\mathfrak{K} : \mathfrak{O}_{\mathbf{p}_0,0} \to \mathfrak{O}_{\mathbf{p}_1,1}. \tag{107}$$

The idea is that if there are no input or output strands, the functor would be

$$\mathfrak{K} : \mathscr{S} \to \mathscr{S},$$

which is equivalent to specifying its value on S, which is our sl_k-Khovanov stable homotopy type. Similarly as in Sussan [32], we will prove its invariance with respect to the relevant flavor of Reidemeister moves, thereby showing that it is a link invariant.

6.2. **Equivalences of categories.** Let $\mathbf{p} \subseteq gl_n$ be a parabolic with Levi factor ℓ and let

$$\mathfrak{O}_{\mathbf{p},n,k}$$

be the parabolic BGG category of $\mathfrak{O}_{n,k}$ with respect to $\mathbf{p}$. Now consider the parabolic $\mathbf{q} \subseteq gl_{k+n}$ with Levi factor $gl_k \oplus \ell$. Let $\overline{\mathbf{p}}$ also denote the parabolic $\mathbf{b}+\mathbf{p}$ in gl_{k+n} (which has Levi factor $h_k \oplus \ell$) and let $\mathbf{g}_{k,n}$ denote the parabolic in gl_{k+n} with Levi factor $gl_k \oplus gl_n$. Denote by

$$\kappa : \overline{\mathbf{p}} \to \mathbf{q}$$

the inclusion.

We have a canonical projection of Harish-Chandra pairs

$$\pi : (\mathscr{O}_{H_k \times L}, \mathbf{g}_{k,n}) \to (\mathscr{O}_{H_k \times L}, gl_k \oplus gl_n).$$

Denote by Δ the power of the determinant comodule of $\mathscr{O}_{GL_k}$ with weight equal to the difference of the (ρ')'s of gl_{n+k} and gl_k. (Note: all these Lie algebras are associated with reflexive transitive relations, and hence have versions over S.) Let V denote the $(\mathscr{O}_{H_k}, gl_k)$-co-Verma module on the character with ρ'-shifted weight $(k-1, k-2, \dots, 0)$. Let

$$f : (\mathscr{O}_{H_k \times L}, \mathbf{g}_{k,n}) \to (\mathscr{O}_{H_k \times L}, gl_{k+n})$$

be the canonical "inclusion" of Harish-Chandra pairs. Consider the functor

$$\psi : f_* \circ \pi^* \circ ((\Delta \wedge V) \underline{\wedge} ?) : \mathfrak{O}_{\mathbf{p},n,k} \to \mathfrak{O}_{\overline{\mathbf{p}},k+n,k} \tag{108}$$

where $\underline{\wedge}$ denotes the "external smash product" which from representations of the Harish-Chandra pairs $(\mathscr{O}_{H_k}, gl_k)$ and $(\mathscr{O}_L, gl_n)$ produces a representation of $(\mathscr{O}_{H_k \times L}, gl_k \oplus gl_n)$.

Now denote

$$\phi = \kappa_* \circ \psi : \mathfrak{O}_{\mathbf{p},n,k} \to \mathfrak{O}_{\mathbf{q},k+n,k}. \tag{109}$$

Lemma 29. *The functor ϕ induces an equivalence of derived categories.*

Proof. Note that by our large prime hypothesis, the functor ϕ defines a bijective correspondence of the co-Verma modules of $\mathbf{p}$ and of $\mathbf{q}$ in the given weight range. In the given weight range, the functor further induces an isomorphism of Hom-sets over $\mathbb{Z}_p$, and over S_p they will just be tensored with the stable p-stems. □

Comment: The equivalence of derived categories ϕ is actually dual to the equivalence used by Sussan [32] in the sense that we use right instead of left adjoints. The reason for this variation is that we have developed the right adjoint functor f_* to f^* for a morphism of Harish-Chandra pairs, so using the right adjoints throughout is easier. The constructions and observations which follow actually do not depend on any special properties of this equivalence of categories.

Now using this, following Sussan [32], for a sequence of numbers $n_1, \dots, n_m$, $1 \leq n_i \leq k$, letting $n = \sum n_i$ and letting n' be the sum of all the n_i's which are not equal to k, and letting $\mathbf{p}$ be the parabolic in gl_n with Levi factor

$$gl_{n_1} \oplus \cdots \oplus gl_{n_m} \tag{110}$$

and letting $\mathbf{p\prime}$ be the parabolic in $gl_{n'}$ with Levi factor (110) modified by omitting all gl_k summands, we shall construct an equivalence of categories

$$\mathfrak{O}_{\mathbf{p},n,k} \to \mathfrak{O}_{\mathbf{p}',n',k}. \tag{111}$$

The equivalence (111) is constructed in [32] by combining sums of equivalences of derived categories of the form

$$\mathfrak{O}_{\mathbf{p}_1,j+k,k} \to \mathfrak{O}_{\mathbf{p}_2,k+j,k} \tag{112}$$

where $\mathbf{p}_1$ resp. $\mathbf{p}_2$ is the parabolic with Levi factor $gl_j \oplus gl_k$ resp. $gl_k \oplus gl_j$, followed by the inverse of the equivalence of Lemma 29. We follow the same approach. The equivalence (112) is constructed by considering the parabolic $\mathbf{p}$ with Levi factor $gl_j \oplus gl_{k-j} \oplus gl_j$ and composing the κ^* with respect to $\mathbf{p}_1$, $\mathbf{p}$ with the $\kappa_\sharp$ with respect to $\mathbf{p}$, $\mathbf{p}_2$. Again, (112) is then a derived equivalence by the large prime hypothesis.

6.3. **Defining the Khovanov cube.** In what follows, we shall use κ generically for the type of "inclusion" of Harish-Chandra pairs which occurs in Theorem 28, when there is no danger of confusion. When more than one such morphism is in sight, we shall be more specific about the notation.

Now for a projection of a tangle T as above, assume we have assigned a functor (107) to T. Now assume that $n(s)$ and $n(s+1)$ in the sum defining n_{a_1} are 1 and $k-1$ (in either order) and assume a tangle projection T' is obtained by replacing a_1 with $a_1+\epsilon$ and joining the s'th and $(s+1)$'st strands at a_1. Then the functor (107) for the tangle T' is obtained by taking the functor (107) for the tangle T, then applying the derived left Zuckermann from $\mathbf{p}_1$ to $\mathbf{p}_1'$ (where $\mathbf{p}_1'$ is the corresponding parabolic with respect to T'), followed by the equivalence (111).

Symmetrically, if $n(s)$ and $n(s+1)$ in the sum defining n_{a_0} are 1 and $k-1$ (in either order) and assume a tangle projection T' is obtained by replacing a_0 with $a_0-\epsilon$ and joining the s'th and $(s+1)$'st strands at a_0. Then the functor (107) for the tangle T' is obtained by applying the inverse of the equivalence (111), followed by the κ^* functor with respect to $\mathbf{p}_0$ and $\mathbf{p}_0'$, followed by the functor (107) for T.

Now assume that $n(s)=n(s+1)=1$ in the sum defining, say, n_{a_1} for an oriented tangle projection T as above and suppose that a tangle projection T' is obtained by replacing a_1 with $a_1+\epsilon$, and adding a crossing between the s'th and $(s+1)$'st strand. Consider the parabolic $\mathbf{p}_1'$ whose Levi factor is generated by the Levi factor of $\mathbf{p}_1$ and a copy of gl_2 on the s'th and $(s+1)$'st strands.

If the s'th strand at a_1 of T crosses above the $(s+1)$'st strand, then assign to T' the functor (107) obtained from the functor (107) for T followed by the homotopy cofiber of the counit of adjunction

$$\kappa^*\kappa_* \to Id \tag{113}$$

where κ is with respect to the parabolics $\mathbf{p}_1$, $\mathbf{p}_1'$.

If the $(s+1)$'st strand of T at a_1 crosses above the s'th strand, then assign to T' the functor (107) obtained from the functor (107) for T followed by the homotopy cofiber of the unit of adjunction

$$Id \to \kappa^*\kappa_\sharp \tag{114}$$

where again, κ is with respect to $\mathbf{p}_1$ and $\mathbf{p}_1'$.

Now completely analogously as in Sussan [32], the Reidemeister relations follow from Diagram relations 1-5 of [32]. The remainder of this paper consists of proving these relations.

6.4. **Diagram relation 2.** The statement we need is contained in the following

Theorem 30. *Let* $\mathbf{p} \subset \mathbf{q} \subseteq gl_n$ *be parabolics where* $\mathbf{p}$ *has Levi factor* $\ell = \ell_1 \oplus gl_1 \oplus gl_1 \oplus \ell_2$ *and* $\mathbf{q}$ *has Levi factor* $m = \ell_1 \oplus gl_2 \oplus \ell_2$. *Then the functor*

$$\kappa^* : \mathfrak{O}_{\mathbf{q},n,k} \to \mathfrak{O}_{\mathbf{p},n,k}$$
$$\kappa_*, \kappa_\sharp : \mathfrak{O}_{\mathbf{p},n,k} \to \mathfrak{O}_{\mathbf{q},n,k}$$

satisfy

$$R\kappa_* \cong L\kappa_\sharp[-2], \tag{115}$$

$$L\kappa_\sharp\kappa^* \cong Id \vee Id[2]. \tag{116}$$

Proof. In fact, (115) is a special case of Theorem 28. For (116), we can think of this is an example of Verdier duality in the case of Kan extensions. We need to find an invertible object ω in $D\mathfrak{O}_{\mathbf{p},n,k}$ and a morphism in $\mathfrak{O}_{\mathbf{q},n,k}$ of the form

$$t : S \to \kappa_\sharp\omega^{-1}. \tag{117}$$

The duality morphism is then the morphism in the derived category induced by

$$\begin{tikzcd}
\kappa_*(X \wedge \omega) \arrow[d, "t\wedge Id"] \\
\kappa_\sharp\omega^{-1} \wedge \kappa_*(X \wedge \omega) \arrow[d, "\varrho"] \\
\kappa_\sharp(\omega^{-1} \wedge \kappa^*\kappa_*(X \wedge \omega)) \arrow[d, "\kappa_\sharp(Id\wedge\epsilon)"] \\
\kappa_\sharp(\omega^{-1} \wedge X \wedge \omega) \arrow[d, "\simeq"] \\
\kappa_\sharp(X)
\end{tikzcd} \tag{118}$$

where ϱ is the projection formula equivalence

$$(\kappa_\sharp X) \wedge Y \simeq \kappa_\sharp(X \wedge \kappa^* Y). \tag{119}$$

To prove a duality, we must choose ω and (117) so that (118) induces an isomorphism in the derived category. In the present case, S is the "trivial representation" and $\omega = S[2]$. In the present case, the construction of (117) is an explicit calculation, as is (118) in the case of parabolic co-Verma modules. This proves (115).

To prove (116), apply the projection formula

$$X \wedge \kappa_\sharp\omega^{-1} \simeq \kappa_\sharp(\kappa^* X \wedge \omega^{-1}),$$

and calculate more precisely

$$\kappa_\sharp \omega^{-1} \simeq S \vee S[-2].$$

□

6.5. **Diagram relation 1.** The required statement in this case is

Theorem 31. *Let* $\mathbf{p} \subset \mathbf{q} \subseteq gl_n$ *be parabolics where* $\mathbf{p}$ *has Levi factor* $\ell = \ell_1 \oplus gl_i \oplus gl_j \oplus \ell_2$ *where* $\{i, j\} = \{1, k-1\}$ *and* $\mathbf{q}$ *has Levi factor* $m = \ell_1 \oplus gl_k \oplus \ell_2$. *Then the functors*

$$\kappa^* : \mathfrak{O}_{\mathbf{q},n,k} \to \mathfrak{O}_{\mathbf{p},n,k}$$

$$\kappa_*, \kappa_\sharp : \mathfrak{O}_{\mathbf{p},n,k} \to \mathfrak{O}_{\mathbf{q},n,k}$$

satisfy

$$R\kappa_* \cong L\kappa_\sharp[-2k+2], \tag{120}$$

$$L\kappa_\sharp \kappa^* \cong Id \vee Id[2] \vee \cdots \vee Id[2k-2]. \tag{121}$$

Proof. Analogous to the proof of Theorem 30 with the exception that

$$\omega = S[2k-2],$$

and one calculates

$$\kappa_\sharp \omega^{-1} = S \vee S[-2] \vee \cdots \vee S[-2k+2].$$

□

6.6. **Diagram relation 3.**

Theorem 32. *Let*

$$\mathbf{p} \supset \mathbf{q} \subset \mathbf{p}' \tag{122}$$

be parabolics in gl_n *with Levi factors* $\ell_1 \oplus gl_1 \oplus gl_k \oplus \ell_2$, $\ell_1 \oplus gl_1 \oplus gl_1 \oplus gl_{k-1} \oplus \ell_2$, $\ell_1 \oplus gl_2 \oplus gl_{k-1} \oplus \ell_2$, *respectively. Denote the first inclusion (122) by* κ *and the second inclusion (122) by* ν. *Then*

$$L\kappa_\sharp \nu^* L\nu_\sharp \kappa^* \cong Id[2] \vee Id[4] \vee \cdots \vee Id[2k-2]. \tag{123}$$

Proof. Using Theorem 31, we have a unit of adjunction

$$Id[0] \vee \cdots \vee Id[2k] \simeq L\kappa_\sharp \kappa^* \to L\kappa_\sharp \nu^* L\nu_\sharp \kappa^*.$$

Restrict this morphism to

$$Id[2] \vee Id[4] \vee \cdots \vee Id[2k-2].$$

One verifies [32] that the left derived functor of the morphism obtained is an equivalence. □

6.7. **Diagram relation 4.** Consider the diagram of parabolics in gl_n:

$$\begin{tikzcd}
\mathbf{p}' & \mathbf{p} \arrow[l, "\supset", "\kappa"'] \\
\mathbf{q} \arrow[ur, "\nu", "\subset"'] \arrow[r, "\tau", "\subset"'] \arrow[d, "\sigma"', "\subset"] & \mathbf{q}' \\
\mathbf{s} &
\end{tikzcd}$$

which on Levi factors is

$$\begin{tikzcd}
& \ell_1 \oplus gl_k \oplus gl_k \oplus \ell_2 & \ell_1 \oplus gl_k \oplus gl_1 \oplus gl_{k-1} \oplus \ell_2 \arrow[l, "\supset"] \\
\ell_1 \oplus gl_{k-1} \oplus gl_1 \oplus gl_1 \oplus gl_{k-1} \oplus \ell_2 \arrow[urr, "\subset"'] \arrow[rr, "\subset"'] \arrow[d, "\subset"'] & & \ell_1 \oplus gl_{k-1} \oplus gl_1 \oplus gl_k \oplus \ell_2 \\
\ell_1 \oplus gl_{k-1} \oplus gl_2 \oplus gl_{k-1} \oplus \ell_2. & &
\end{tikzcd}$$

Theorem 33. *There is an isomorphism in the derived category*

$$L\nu_\sharp\sigma^*L\sigma_\sharp\tau^*L\tau_\sharp\sigma^*L\sigma_\sharp\nu^* \cong Id[2k]\vee\kappa^*L\kappa_\sharp[4]\vee\cdots\vee\kappa^*L\kappa_\sharp[2k-2]. \tag{124}$$

Proof. We have the unit of adjunction

$$\begin{tikzcd}
Id \arrow[d, "\eta"] \\
R\nu_*\sigma^*R\sigma_*\tau^*L\tau_\sharp\sigma^*L\sigma_\sharp\nu^* \arrow[d, "\cong"] \\
L\nu_\sharp\sigma^*L\sigma_\sharp\tau^*L\tau_\sharp\sigma^*L\sigma_\sharp\nu^*[-2k]
\end{tikzcd} \tag{125}$$

where the bottom arrow follows from Theorems 31, 30. Similarly, we have morphisms

$$\begin{tikzcd}
\kappa^*L\kappa_\sharp[0]\vee\cdots\vee\kappa^*L\kappa_\sharp[2k-2] \arrow[d, "\cong"] \\
L\nu_\sharp\kappa^*L\kappa_\sharp\nu^* \arrow[d, "L\nu_\sharp\eta\kappa^*L\kappa_\sharp\eta\nu^*"] \\
L\nu_\sharp\sigma^*L\sigma_\sharp\tau^*L\tau_\sharp\sigma^*L\sigma_\sharp\nu^*.
\end{tikzcd} \tag{126}$$

(Note that the left derived pushforwards and pullback associated with κ commute with the left derived pushforwards and pullbacks associated with ν.)

Now restricting (125) and (126) to the summands which occur on the right hand side of (124) gives the required isomorphism in the derived category. The fact that it is an isomorphism is verified on co-Verma modules as in [32]. ◻

6.8. **Diagram relation 5.** First consider a diagram of inclusions of parabolic subalgebras of gl_n of the form

$$\begin{tikzcd} \mathbf{p}_1 \arrow[r, "\subset", "\nu_1"'] & \mathbf{q} \\ \mathbf{p} \arrow[u, "\subset", "\kappa_1"'] \arrow[r, "\subset", "\kappa_2"'] & \mathbf{p}_2 \arrow[u, "\subset", "\nu_2"'] \end{tikzcd}$$

with corresponding inclusions of Levi factors

$$\begin{tikzcd} \ell_1 \oplus gl_2 \oplus gl_1 \oplus \ell_2 \arrow[r, "\subset"] & \ell_1 \oplus gl_3 \oplus \ell_2 \\ \ell_1 \oplus gl_1 \oplus gl_1 \oplus gl_1 \oplus \ell_2 \arrow[u, "\subset"] \arrow[r, "\subset"] & \ell_1 \oplus gl_1 \oplus gl_2 \oplus \ell_2. \arrow[u, "\subset"] \end{tikzcd}$$

Lemma 34. *In the derived category, we have an isomorphism*

$$L(\kappa_1)_\sharp \kappa_2^* L(\kappa_2)_\sharp \kappa_1^* \cong Id[2] \vee \nu_1^* L(\nu_1)_\sharp. \tag{127}$$

Proof. The morphism from the left hand side of (127) to the first summand on the right hand side is the composition

$$\begin{tikzcd} L(\kappa_1)_\sharp \kappa_2^* (L\kappa_2)_\sharp \kappa_1^*[-2] \arrow[d, "\cong"] \\ L(\kappa_1)_\sharp \kappa_2^* (R\kappa_2)_* \kappa_1^* \arrow[d, "\epsilon"] \\ Id \end{tikzcd} \tag{128}$$

where the top equivalence is by Theorem 30 and the bottom arrow is the counit of adjunction.

To construct a morphism from the left hand side of (127) to the second summand of the right hand side, we proceed in several steps. First, we have a counit of adjunction

$$L(\kappa_1)_\sharp \kappa_1^* \to Id, \tag{129}$$

so by composition, we obtain a morphism

$$L(\nu_2)_\sharp L(\kappa_2)_\sharp \kappa_1^* = L(\nu_1)_\sharp L(\kappa_1)_\sharp \kappa_1^* \to L(\nu_1)_\sharp. \tag{130}$$

Again, by composition, we obtain a morphism

$$L(\kappa_2)_\sharp \kappa_1^* \to \nu_2^* L(\nu_2)_\sharp L(\kappa_2)_\sharp \kappa_1^* \to \nu_2^* L(\nu_1)_\sharp \tag{131}$$

where the first morphism is given by a unit of adjunction. By composition, again, we obtain a morphism

$$\kappa_2^* L(\kappa_2)_\sharp \kappa_1^* \to \kappa_2^* \nu_2^* L(\nu_1)_\sharp = \kappa_1^* \nu_1^* L(\nu_1)_\sharp. \tag{132}$$

By another composition, we then obtain a morphism

$$L(\kappa_1)_\sharp \kappa_2^* L(\kappa_2)_\sharp \kappa_1^* \to L(\kappa_1)_\sharp \kappa_1^* \nu_1^* L(\nu_1)_\sharp \to \nu_1^* L(\nu_1)_\sharp \tag{133}$$

which is a morphism from the left hand side of (127) to the second summand of the right hand side, as needed. Summing with (128), we obtain a morphism from the left hand side to the right hand side of (127), which is checked to be an isomorphism in the derived category by calculation on co-Verma modules. □

The diagram relation 5 is expressed by the following

Theorem 35. *We have an isomorphism in the derived category*

$$\begin{gathered}\kappa_1^* L(\kappa_1)_\sharp \kappa_2^* L(\kappa_2)_\sharp \kappa_1^* L(\kappa_1)_\sharp \vee \kappa_2^* L(\kappa_2)_\sharp[2] \cong \\ \kappa_2^* L(\kappa_2)_\sharp \kappa_1^* L(\kappa_1)_\sharp \kappa_2^* L(\kappa_2)_\sharp \vee \kappa_1^* L(\kappa_1)_\sharp[2].\end{gathered} \tag{134}$$

Proof. By Lemma 34, we have isomorphisms in the derived category

$$\begin{gathered}\kappa_1^* L(\kappa_1)_\sharp \kappa_2^* L(\kappa_2)_\sharp \kappa_1^* L(\kappa_1)_\sharp \vee \kappa_2^* L(\kappa_2)_\sharp[2] \cong \\ \kappa_1^* \nu_1^* L(\nu_1)_\sharp L(\kappa_1)_\sharp \vee \kappa_1^* L(\kappa_1)_\sharp[2] \vee \kappa_2^* L(\kappa_2)_\sharp[2] \cong \\ \kappa_2^* \nu_2^* L(\nu_2)_\sharp L(\kappa_2)_\sharp \vee \kappa_1^* L(\kappa_1)_\sharp[2] \vee \kappa_2^* L(\kappa_2)_\sharp[2] \cong \\ \kappa_2^* L(\kappa_2)_\sharp \kappa_1^* L(\kappa_1)_\sharp \kappa_2^* L(\kappa_2)_\sharp \vee \kappa_1^* L(\kappa_1)_\sharp[2]\end{gathered}$$

as claimed. □

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