D-STRUCTURES AND DERIVED KOSZUL DUALITY FOR UNITAL OPERAD ALGEBRAS

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Abstract. Generalizing a concept of Lipshitz, Ozsváth and Thurston from Bordered Floer homology, we define D-structures on algebras of unital operads. This construction gives rise to an equivalence of derived categories, which can be thought of as a unital version of Koszul duality using non-unital Quillen homology, even though the non-unital Quillen homology of unital objects is zero.

1. Introduction

Koszul duality for operads and algebras over them was introduced in the landmark paper [5] by Ginzburg and Kapranov. For example, as long as we work over a field of characteristic 0, the operads encoding commutative and Lie algebras are Koszul dual, while the operad encoding associative algebras is self-dual. The Koszul duality of Ginzburg and Kapranov [5] has both a non-derived and a derived version. The derived version involves a kind of a bar construction on an operad, which we call the *Ginzburg-Kapranov bar construction*. The Ginzburg-Kapranov bar construction on an operad is a differential graded co-operad, which, under suitable finiteness obstructions, can be dualized to a differential graded operad, the *derived Koszul dual* of the original operad. When there is not enough finiteness to dualize, the DG co-operad can also be used directly, and we refer to it as the *Koszul transform* of the original operad. Just as in the more classical Koszul duality of associative algebras (cf. [20]), there is a property of an operad being *Koszul* which implies that the Koszul dual operad is, in fact, quasi-isomorphic to a (non-differential graded) operad, which is then called the *non-derived Koszul dual*.

The concept of derived Koszul duality is (as always) more important to homotopy theory foundations than the corresponding non-derived concept, although the non-derived concept is a useful calculational tool. Derived Koszul dual operads are a part of a more general scheme, which, in some sense goes back to Quillen [21]: In any based category C with finite products, we may define the category of abelian objects

$Ab(C)$. The forgetful functor

$$U : Ab(C) \to C$$

has a left adjoint L, called *abelianization*. The functor LU is then a co-monad in $Ab(C)$. In the presence of a mechanism creating derived categories, if we denote by M the derived version of the co-monad LU, then the Koszul transform of the derived category DC should be the category of M-co-algebras. This is, of course, somewhat vague, and it gets even more so: Abelian objects may be replaced by E_∞-objects, and ultimately, the functor L may be replaced by Dwyer-Kan stabilization (an anague of the topological notion of spectra in a based category with certain additional structure, cf. [3]). It is very difficult to get any precise theorems along these lines in the most general case because of convergence problems. Nevertheless, it was observed by Kontsevich that derived Koszul duality for operads is a part of this general scenario (see [11, 12, 10, 9]).

One of the puzzles of Koszul duality has been that it does not seem to interact well with units. The main problem is that unital (non-augmented) operads, algebras etc. do not tend to form a based category, and therefore the scheme described in the last paragraph does not work directly. Furthermore, if one takes Quillen cohomology of unital objects in the corresponding non-unital category, one usually gets zero ([21]). It has been an open question whether there is a version of Koszul duality which works, say, for unital algebras over unital operads, in a sense which would generalize the above categorical scheme.

The main result of the present paper is to define derived Koszul duality for unital algebras over unital operads. Our approach is to take non-unital homology of the unital $\mathcal{C}$-algebra, which by [11, 12] is calculated by the Ginzburg-Kapranov bar construction. Even though these bar constructions have 0 chain homology, they have however additional structure (which we call *D-structure*). There is a natural way of defining morphisms and equivalences of D-structures, and prove that the resulting derived category is equivalent to the category of $\mathcal{C}$-algebras. Comment: while we do rigorously construct the derived category of D-structures for a general unital operad, we do not construct a Quillen model structure [4] for them. Constructing such a structure remains an interesting open problem.

The term D-structure is taken from the *bordered Floer homology* of Lipshitz, Oszváth and Thurston [14], which is a (pointed) topological quantum field theory built out of the Heegaard-Floer homology invented by Ozsváth and Szabó [18]. From the point of view of the present paper, they use our derived Koszul duality in the special case of A-modules where A is a unital algebra: both the modules and their Koszul transform D-structures occur naturally as combinatorial objects calculating Bordered Floer homology. There are a few technical nuances, for example, on one side, [14] consider A_∞-modules instead of strict modules, and also consider the differential as one of the operations. We replace this by a context most suitable to our techniques; the different variants of the concepts, both on the level of modules and D-structure, are easily seen to lead to equivalent derived categories. In any case, it is very important to [14] to have an *equivalence of derived categories*; they need to construct a pairing of two objects in a certain geometrically given category, while a natural geometrically given pairing is between an object of the category and an object of its Koszul transform. We do reproduce a full generalization of this equivalence of categories to the general context of unital algebras over unital operads.

A few technical comments are in order. First, fixing our context: Ginzburg and Kapranov [5] work over a field of characteristic 0, while Lipshitz, Ozsváth and Thurston [14] work over a field of characteristic 2. The motivation of [14] is to avoid the discussion of signs. In this paper, we do work out the signs, and in a sense, this is one of our key points: While in [5], a clever exterior algebra method is introduced to control the signs in the Ginzburg-Kapranov bar construction, we need to extend this construction by a *Clifford algebra* to handle the differential graded case. This becomes important because the derived Koszul transform of a DG operad is, in fact, naturally *bigraded*. We need to totalize this into a singly graded chain complex. While there are standard signs for totalizing double chain complexes, these signs *do not work for generalizing the D-structures of Lipshitz, Ozsváth and Thurston* [14] *outside of characteristic* 2! The Clifford algebra method we use reconciles the signs of the internal grading with the Ginzburg-Kapranov bar construction completely, and works in the setting we need.

The issue of signs is related to the more general point that while we are interested in derived categories, D-structure is a point-set level algebraic structure, and therefore for our purpose, objects cannot be freely replaced by quasiisomorphic ones. This is one of the reasons why we must use the Ginzburg-Kapranov bar construction [5]; we do not know how to make our theory replacing it with a more standard construction, such as the two-sided bar construction [16], (which, of course, could be used if we worked only up to homotopy). In some sense, [5] deal with the same issue in their original concept of non-unital Koszul duality.

Another point is that the reason Ginzburg-Kapranov [5] work in characteristic 0 is that otherwise the monad associated with an operad does not preserve quasiisomorphisms. This happens in some very basic cases, for example for commutative algebras. This is a very well recognized phenomenon which plays an important role in homotopy theory (cf. [17], or, in an algebraic context, [13]). In this paper, we work in the category of modules over an arbitrary field, and impose a condition we call Σ-cofibrancy on the operad. This condition also comes from [17, 13], and has of course been since used by other authors, too. It is automatically satisfied in characteristic 0.

Finally, we would like to comment on the relatinoship of our results with previous work, in particular the recent results of Hirsch and Milles [8]. The main concept of [8] is that of a *properad*, which is a variant of previous notions of PROPs [22] and dioperads [6]. PROPs, like operads, were first introduced in the context of infinite loop space theory [2]. To relate results to our work, it is important to note that [8] establish Koszul duality for properads, but not *algebras* over them. For example, the forgetful functors from properad algebra categories to modules over the base field are typically neither right or left adjoints, so it is not clear what Hochschild or Quillen homology would even mean for them. On the other hand, properads (or dioperads) themselves are algebras over appropriate multisorted operads. While in this paper we do not discuss the multisorted case, our statements generalize easily to that context. A derived version of the concept of *curvature* of [8] is then equivalent to a special case of (a multisorted version of) D-structure, although the definition simplifies somewhat due to the "monoidal" nature of properads. Since this case is not of special interest to us, we do not go into details.

The present paper is organized as follows: Because of the delicate technical nature of our construction, and the presence of a large number of variants of similar concepts in the literature, we found it necessary to make the present paper as self-contained as possible, even at the cost of redefining some known concepts, when there is ambiguity in them. In Section 2, we treat these necessary technical prerequisites. In Section 3, we treat rigorously the notion of a *tree*. For us, trees are ordered, or "planar" trees. Because of the sign issue, which is central to us, we also choose to be more pedantic and rigorous than is customary in this context. In Section 4, we review the Ginzburg-Kapranov bar construction in the setting we need, and implement the relevant sign devices. In Section 5, we introduce our version of the concept of a D-structure. In Section 6, we construct the derived category of D-structures, and prove that it is equivalent to the Quillen derived category of algebras over the original operad.

2. Preliminaries

In this paper, we will work with unital operads $\mathcal{C}$ in the category of chain complexes of K-modules where K is a field. We will also call them *DG K-module operads.* This means a sequence $\mathcal{C}(n)$ of chain complexes of K-modules, together with an action of the permutation group Σ_n on $\mathcal{C}(n)$, a unit chain map

$$K \to \mathcal{C}(1),$$

and an operation

$$\mathcal{C}(n_1) \otimes \cdots \otimes \mathcal{C}(n_k) \otimes \mathcal{C}(k) \to \mathcal{C}(n_1 + \cdots + n_k), \tag{1}$$

$$x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \mapsto \gamma(x_1, \ldots, x_k; x_{k+1})$$

satisfying the usual equivariance, associativity and unit axioms modelled on the example $\mathcal{H}_M(n) = Hom(M^{\otimes n}, M)$ where M is a chain complex of Kmodules, and Hom denotes the internal Hom in the category of chain complexes of K-modules ([15]). As usual, when dealing with graded objects, we apply the Koszul sign

$$(-1)^{jk}$$

in the switch homomorphism between $x \otimes y$ and $y \otimes x$ for homogeneous elements x, y of degrees j, k, respectively.

It may be more common to put the $\mathcal{C}(k)$ term first in the tensor product (1). We chose the current order of variables because we work in the context of trees, which we visualize as having roots in the bottom: When writing writing the trees in one line, it seems natural to write the upper parts of the tree to the left and the root to the right. It is, obviously, only a matter of signs.

It is also useful to introduce the operation

$$\gamma_j : \mathcal{C}(k) \otimes \mathcal{C}(n) \to \mathcal{C}(n + k - 1)$$

given by

$$\gamma_j(x, y) = \gamma(\underbrace{1, \ldots, 1}_{j-1 \text{ times}}, x, \underbrace{1, \ldots, 1}_{k-j \text{ times}}).$$

When considering non-unital operads (which we do not do in this paper, but which is, for example, the basic setup of [5]), one usually does include the operations γ_j in the definition: the operation γ can be recovered from them, but not vice versa.

A morphism of operads is a chain map which preserves the operations γ, the unit and the Σ_n-equivariances. By a *DG-$\mathcal{C}$-algebra* A we mean a homomorphism of operads

$$\mathcal{C} \to \mathcal{H}_A.$$

This is equivalent data to having operations

$$\theta : \underbrace{A \otimes \cdots \otimes A}_{n \text{ times}} \otimes \mathcal{C}(n) \to A$$

which satisfy obvious associativity, unit and equivariance conditions. Morphisms of $\mathcal{C}$-algebras are chain maps which preserve all the operations.

We will also use the notion of a monad, which is a generalization of an operad. A monad in a category Q is a functor $C : Q \to Q$ together with a product $\mu : CC \to C$ and a unit $\eta : Id \to C$ which satisfy associativity and unit axioms. A *C-algebra* X consists of a natural transformation $\theta : CX \to X$ which satisfies an associativity with respect to μ and a unit axiom with respect to η. Morphisms of C-algebras are morphisms in C which commute with the operation θ.

For an operad $\mathcal{C}$, there is a canonical monad C such that $\mathcal{C}$-algebras are the same as C-algebras (they form canonically equivalent categories). The monad is given by

$$CX = \bigoplus_{n \geq 0} X^{\otimes n} \otimes_{\Sigma_n} \mathcal{C}(n) \tag{2}$$

where the symmetric group acts on $X^{\otimes n}$ by permutations (with the Koszul signs).

In the category of chain complexes of K-modules, by an *equivalence* we mean a chain map which induces an isomorphism in homology. If X is a chain complex, $X[n]$ denotes the chain complex X with dimensions shifted by n: $X[n]_k = X_{k-n}$.

In important point is that for a general operad, when $f : X \to Y$ is an equivalence, $Cf : CX \to CY$ may not be an equivalence. In fact, this has less to do with the notion of an operad than with structure of chain complexes of $K[\Sigma_n]$-modules. Let $I[\Sigma_n]$ denote the chain complex of $K[\Sigma_n]$-modules where

$$\begin{array}{rll} I[\Sigma_n]_i = & K[\Sigma_n] & \text{for } i = 0, 1 \\ & 0 & \text{else,} \end{array}$$

where the differential is the identity. Then we have an obvious chain map of $K[\Sigma_n]$-modules $\epsilon : K[\Sigma_n] \to I[\Sigma_n]$. (An ungraded module, when considered graded without further specification, is, as usual, considered to be in dimension 0). A chain complex of $K[\Sigma_n]$-modules X is called *cell* if

$$X \cong \lim_{n} X_{(n)}$$

where $X_{(-1)} = 0$ and there exist sets I_n, maps $q_n : I_n \to \mathbb{Z}$, and chain maps of $K[\Sigma_n]$-modules

$$f_n : \bigoplus_{i \in I_n} K[\Sigma_n][q_n(i)] \to X_{(n-1)}$$

such that $X_{(n)}$ is a pushout of the diagram

$$\begin{array}{ccc} \bigoplus_{i \in I_n} K[\Sigma_n][q_n(i)] & \xrightarrow{f_n} & X_{(n-1)} \\ \Big\downarrow{\scriptstyle \bigoplus \epsilon[q_n(i)]} & & \\ \bigoplus_{i \in I_n} I[\Sigma_n][q_n(i)] & & \end{array}$$

We will call a chain complex of $K[\Sigma_n]$-modules *S-cofibrant* if it is a retract of a cell chain complex of $K[\sigma_n]$-modules. (Note that when K is a field of characteristic 0, the assumption is automatically satisfied and hence vacuos.) We will call an DG K-module operad $\mathcal{C}$ *S-cofibrant* if each $\mathcal{C}(n)$ is an S-cofibrant chain complex of $K[\Sigma_n]$-modules. The main point of considering these notions is the following

Proposition 1. *Let X be an S-cofibrant chain complex of $K[\Sigma_n]$-modules. Then the functor*

$$X \otimes_{K[\Sigma_n]} ? : K[\Sigma_n]\text{-}modules \to K[\Sigma_n]\text{-}modules$$

preserves equivalences. Consequently, if $\mathcal{C}$ is an S-cofibrant DG K-module operad, then the associated monad

$$C : \textit{chain complexes of } K\textit{-modules} \to \textit{chain complexes of } K\textit{-modules}$$

preserves equivalences.

3. Trees

In this section, we will rigorously define, and describe basic operations on what is usually referred to as "planar trees". Roughly speaking, they are (finite) rooted trees where the set of vertices which are immediately above each given vertex (i.e. are attached to it by an edge) come with a specified linear ordering. This determines a linear ordering on the entire set of vertices of the tree. In the context of the present paper, where there is need for extra sensitivity regarding signs, we felt compelled to be perhaps more rigorous about this concept than is usually customary, and write everything down "algebraically", without using to pictures. In connection with this, we should, of course, note that mild variations in the concept are possible, for example, the ordering could be reversed. One variation which could be considered more substantial is that we distinguish between "leaves" and "non-leaf vertices of valency 0". This is because of the fact that we work with operads $\mathcal{C}$ and $\mathcal{C}$-algebras A where we do not require $\mathcal{C}(0) = 0$: elements of $\mathcal{C}(0)$ are then attached to non-leaf vertices of valency 0, while elements of a $\mathcal{C}$-algebra A are attached to leaves.

Denote $\mathbf{n} = \{1, \dots, n\}$, $|S|$ for a (finite) set S will denote the cardinality of S.

Definition 2. In this paper, a *tree* (n, s, L) is the following data: a subset $L \subseteq \mathbf{n}$ (called the set of *leaves*) and map

$$\mathbf{n} - \mathbf{1} \; (= \mathbf{n} \smallsetminus \{n\}) \rightarrow \mathbf{n} \smallsetminus L$$

such that

(1) $s(x) > x$
(2) $x \leq y < s(x) \Rightarrow s(y) \leq s(x)$.

For $i \in \mathbf{n}$, the number $v_s(i) = |s^{-1}(i)|$ will be called the *valency* of i. Therefore, for $i \in L$, we have $v(i) = 0$.

To interpret this definition in terms of usual planar trees, every x is connected to $s(x)$ by an edge; the vertex $s(x)$ is "below" the vertex x in the planar tree. Therefore, the root is the greatest element, n.

Lemma 3. *Let (n, s, L) be a tree, $n \notin L$ (note that $n \in L$ is only possible if $n = 0$) and let*

$$\{k_1 < k_2 < \cdots < k_m\} = s^{-1}(n).$$

Then we have $k_m = n - 1$. Putting $k_0 = 0$, $n_i = k_i - k_{i-1}$, $i = 1, \dots, m$, (n_i, s_i, L_i) are trees where

$$s_i(j) = s(j + k_{i-1}) - k_{i-1},$$
$$L_i = \{x - k_{i-1} \mid x \in L\} \cap \mathbf{n_i}.$$

□

We call $(n_i, s_i, L_i)_i$ the sequence of *successor trees* of (n, s, L). Let (n, s, L) be a tree with sequence of successor trees

$$(n_1, s_1, L_1), \dots (n_m, s_m, L_m).$$

Let $\sigma : \mathbf{m} \to \mathbf{m}$ be a permutation. Then, obviously, there is a unique tree (n, s^σ, L^σ) with sequence of successor trees

$$(n_{\sigma(1)}, s_{\sigma(1)}, L_{\sigma(1)}), \ldots, (n_{\sigma(m)}, s_{\sigma(m)}, L_{\sigma(m)}).$$

Let $\sim$ be the smallest equivalence relation on trees such that

(1) $(n, s, L) \sim (n^\sigma, s^\sigma, L^\sigma)$ for any permutation σ applicable.

(2) If (n, s, L), (n', s', L') have sequences of successor trees

$$(n_1, s_1, L_1), \ldots (n_m, s_m, L_m),$$
$$(n_1, s'_1, L'_1), \ldots (n_m, s'_m, L'_m)$$

where $(n_i, s_i, L_i) \sim (n_i, s'_i, L'_i)$, $i = 1, \ldots, m$, then

$$(n, s, L) \sim (n', s', L').$$

Lemma 4. *For trees (n, s, L), (n, s', L'), we have $(n, s, L) \sim (n, s', L')$ if and only if there exists a permutation $\sigma : \mathbf{n} \to \mathbf{n}$ such that $\sigma(L) = L'$ and $s'(\sigma(i)) = \sigma(s(i))$ when applicable.*

□

We call σ the *intertwining permutation* $(n, s, L) \to (n, s', L')$. Note that the permutation σ may not be unique. Trees and intertwining permutations form a groupoid.

Let (n, s, L) be a tree, $n \notin L$ with sequence of successor trees

$$(n_1, s_1, L_1), \ldots, (n_m, s_m, L_m).$$

Choose $j \in \mathbf{m}$ with $n_j \notin L_j$ and let

$$(p_1, t_1, M_1), \ldots (p_q, t_q, M_q)$$

be the sequemce of successor trees of (n_j, s_j, L_j). Then there is a unique tree $(n-1, s^\circ_j, L^\circ_j)$ with sequence of successor trees

$$\begin{aligned}&(n_1, s_1, L_1), \ldots, (n_{j-1}, s_{j-1}, L_{j-1}),\\ &(p_1, t_1, M_1), \ldots, (p_q, t_q, M_q),\\ &(n_{j+1}, s_{j+1}, L_{j+1}), \ldots, (n_m, s_m, L_m).\end{aligned}$$

We define a tree $(n-1, s', L')$ inductively to be an *edge contraction* of a tree (n, s, L) if either

$$(n-1, s', L') = (n-1, s^\circ_j, L^\circ_j)$$

for some j, or (n, s', L') has sequence of successor trees

$$\begin{aligned}&(n_1, s_1, L_1), \ldots, (n_{j-1}, s_{j-1}, L_{j-1}),\\ &(n_j - 1, s'_j, L'_j),\\ &(n_{j+1}, s_{j+1}, L_{j+1}), \ldots, (n_m, s_m, L_m)\end{aligned}$$

where (n_j-1, s'_j, L'_j) is an edge contraction of (n_j, s_j, L_j).

Lemma 5. *For $j \in (\mathbf{n-1}) \smallsetminus L$, there exists a unique edge contraction $(n-1, s', L')$ of the tree (n, s, L) such that the map*

$$\tau = \tau_{s,j} : (\mathbf{n-1}) \to \mathbf{n}$$

given by

$$\tau(k) = \begin{array}{ll} k & \text{for } k < j \\ k+1 & \text{for } k \geq j \end{array}$$

satisfies

$$\tau(s'(k)) = \begin{array}{ll} s\tau(k) & \text{if } s(k) \neq i \\ s(i) & \text{if } s(k) = i. \end{array}$$

Moreover, every edge contraction is obtained in this way. (We call $(n-1, s', L')$ the edge contraction of (n, s, L) at j.*)*

□

We shall also use a particular *left inverse* $\rho = \rho_{s,j}$ of $\tau_{s,j}$ defined by

$$\tau \circ \rho = Id_{\mathbf{n-1}},$$

$$\rho(j) := s(j).$$

A tree (n, s, L) is called a *bush* if $L = \mathbf{n-1}$. A tree (n', s', L') is called a *leaf contraction* of a tree (n, s, L) if there exist $1 \leq i \leq j \leq n$, $j \notin L$, such that

$$s^{-1}(j) = \{i, i+1, \ldots, j-1\} \subseteq L,$$

$n' = n - j + i$ and the map

$$\tau = \tau_{s,i,j} : \mathbf{n'} \to \mathbf{n}$$

given by

$$\tau_{s,i,j}(k) = \begin{array}{ll} k & \text{for } k < i \\ k+j-i & \text{for } k \geq i \end{array}$$

satisfies

$$\tau(s'(k)) = s(\tau(k)) \text{ for all } k \in \mathbf{n'},$$
$$L' = \tau^{-1}(L) \cup \{i\}.$$

We then call (n', s', L') the *leaf contraction of* (n, s, L) *at* (i, j). Again, we shall also use a left inverse $\rho = \rho_{s,i,j}$ of $\tau_{s,i,j}$ where

$$\rho \circ \tau = Id_{\mathbf{n'}},$$

$$\rho(k) := i \text{ for } k \in \{i, i+1, \ldots, j-1\}.$$

Lemma 6. *Let* $\sigma : \mathbf{n} \to \mathbf{n}$ *be an intertwining permutation*

$$\sigma : (n, s, L) \to (n, t, M).$$

1. Let $j \in (\mathbf{n-1}) \smallsetminus L$ *Let* $(n-1, s', L')$ *resp.* $(n-1, t', M')$ *be the edge contractions of* (n, s, L) *resp.* (n, t, M) *at* j *resp.* $\sigma(j)$*. Then the unique permutation*

$$\sigma' : \mathbf{n-1} \to \mathbf{n-1}$$

with

$$\tau_{t,\sigma(j)}\sigma' = \sigma\tau_{s,j}$$

intertwines

$$\sigma' : (n-1, s', L') \to (n-1, t', M').$$

2. Let (n', s', L') *be a leaf contraction of* (n, s, L) *at* (i, j)*. Then there exists a leaf contraction* (n', t', M') *at* $(\sigma(j)-j+i, \sigma(j))$ *and the unique permutation*

$$\sigma' : \mathbf{n}' \to \mathbf{n}'$$

which satisfies

$$\tau_{t,\sigma(j)-k+i}\sigma' = \sigma\tau_{s,i,j}$$

intertwines

$$\sigma' : (n', s', L') \to (n', t', M').$$

□

4. The augmented Ginzburg-Kapranov bar construction

For a tree (n, s, L) and $i \in \mathbf{n}$, denote

$$\epsilon(i) = \epsilon_{(n,s,L)}(i) = |\mathbf{i} \smallsetminus L|.$$

Let K be a field and let $\mathcal{C}$ be an S-cofibrant operad in the category of chain complexes of K-modules. Let A be a graded $\mathcal{C}$-algebra. We let

$$\Lambda(n, s, L)$$

be a K-valued exterior algebra on indeterminates e_i, $i \in \mathbf{n-1} \smallsetminus L$. Let

$$C(n)$$

be a Clifford algebra on indeterminates f_i, $i \in \mathbf{n}$,

$$f_i^2 = 1,\ f_i f_j = -f_j f_i \text{ for } i \neq j.$$

We shall write

$$Det(n, s, L)$$

for the top degree summand of $\Lambda(n,s,L)$, i.e. the sub-K-module of $\Lambda(n,s,L)$ spanned by

$$\prod_{i\in(\mathbf{n-1})\smallsetminus L} e_i.$$

For $\epsilon=(\epsilon_1,\dots,\epsilon_n)\in(\mathbb{Z}/2)^n$, we denote by $C_{n,\epsilon}$ the sub-K-module of C_n spanned by

$$f_1^{\epsilon_1}\dots f_n^{\epsilon_n}.$$

For $\varepsilon\in\mathbb{Z}/2$, and a graded K-module M, denote by M_ε the submodule spanned by all homogeneous elements of degrees equal to $\varepsilon \mod 2$. The *augmented bar construction* $\widetilde{B}_{\mathcal{C}}(A)$ is defined by

$$\begin{array}{l}\widetilde{B}_{\mathcal{C}}(A)=\\ (\displaystyle\bigoplus_{(n,s,L)}\bigoplus_{\epsilon\in(\mathbb{Z}/2)^n} Det(n,s,L)\otimes\\ C_\epsilon\otimes(\displaystyle\bigotimes_{i\in L}A_{\epsilon_i}\otimes\bigotimes_{\substack{i\in\mathbf{n}\smallsetminus L\\ v(i)=k}}\mathcal{C}(k)_{\epsilon_i})[\epsilon_{(n,s,L)}(n)])/\sim .\end{array}$$

The direct sum is over all trees. The equivalence is the smallest congruence of K-modules satisfying, for

$$\sigma:(n,s,L)\to(n,s',L'),$$

$$\prod_i e_i\prod_j f_j\cdot\bigotimes_{i\in L}(x_i\in A)\otimes\bigotimes_{\substack{i\in\mathbf{n}\smallsetminus L\\ v_s(i)=k}}(x_i\in\mathcal{C}(k))$$

$$\sim\prod_i e_{\sigma(i)}\prod_j f_{\sigma(j)}\cdot\bigotimes_{i\in L'}(y_i\in A)\otimes\bigotimes_{\substack{i\in\mathbf{n}\smallsetminus L'\\ v_{s'}(i)=k}}(y_i\in\mathcal{C}(k)).$$

Note: since the algebras $\Lambda(n,s,L)$, C_n are not commutative, the products are taken in the order of indexing.

The differential d on $\widetilde{B}_{\mathcal{C}}(A)$ is the sum of the following kinds of maps:

(1) **The edge contraction summands.** If $(n-1,s',L')$ is an edge contraction of (n,s,L) at j, let

$$\lambda_\rho:\Lambda(n,s,L)\to\Lambda(n-1,s',L')$$

be given by

$$e_i\mapsto e_{\rho(i)},$$

and let

$$c_\rho : C_n \mapsto C_{n-1}$$

be defined by

$$f_j \mapsto f_{\rho(j)}.$$

Then

$$\prod_i e_i \prod_j f_j \cdot \bigotimes_{i\in L}(x_i \in A) \otimes \bigotimes_{\substack{i \in \mathbf{n} \smallsetminus L \\ v_s(i)=k}} (x_i \in \mathcal{C}(k))$$

$$\mapsto \lambda_\rho(\frac{\partial}{\partial e_j} \prod_i e_i) c_\rho(\prod_j f_j) \cdot \bigotimes_{i\in L'}(y_i \in A) \otimes \bigotimes_{\substack{i \in (\mathbf{n-1}) \smallsetminus L' \\ v_{s'}(i)=\ell}} (y_i \in \mathcal{C}(\ell))$$

where

$$y_i = x_{\tau_{s,j}(i)} \text{ if } i \neq s(j)-1,$$

and

$$y_{s(j)-1} = \gamma_\ell(x_j, x_{s_j})$$

if

$$s^{-1}(s(j)) = \{j_1 < \cdots < j_m\},\ j_\ell = j.$$

(2) **The leaf contraction summands.** If (n', s', L') is a leaf contraction of (n, s, L) at (i, j), let, again,

$$\lambda_\rho : \Lambda(n, s, L) \to \Lambda(n', s', L')$$

be given by

$$e_i \mapsto e_{\rho(i)},$$

and let

$$c_\rho : C_n \mapsto C_{n'}$$

be defined by

$$f_j \mapsto f_{\rho(j)}.$$

$$\prod_i e_i \prod_j f_j \cdot \bigotimes_{q\in L}(x_q \in A) \otimes \bigotimes_{\substack{q \in \mathbf{n} \smallsetminus L \\ v_s(q)=k}} (x_q \in \mathcal{C}(k))$$

$$\mapsto \lambda_\rho(\frac{\partial}{\partial e_j} \prod_i e_i) c_\rho(\prod_j f_j) \cdot \bigotimes_{q\in L'}(y_q \in A) \otimes \bigotimes_{\substack{q \in (\mathbf{n'}) \smallsetminus L' \\ v_{s'}(q)=\ell}} (y_q \in \mathcal{C}(\ell))$$

where

$$y_q = x_{\tau_{s,i,j}(q)} \text{ if } q \neq i,$$
$$y_i = \theta(x_i, \ldots, x_{j-1}; x_j).$$

(3) **The internal differential summands.** Denoting by ∂ the internal differential on A and $\mathcal{C}(k)$, and choosing $i \in \mathbf{n}$,

$$\prod_{\ell} e_\ell \prod_j f_j \cdot \bigotimes_{q\in L}(x_q \in A) \otimes \bigotimes_{\substack{q \in \mathbf{n} \smallsetminus L \\ v_s(q)=k}} (x_q \in \mathcal{C}(k))$$

$$\mapsto f_i \prod_{\ell} e_\ell \prod_j f_j \cdot \bigotimes_{q\in L}(y_q \in A) \otimes \bigotimes_{\substack{q \in (\mathbf{n}) \smallsetminus L \\ v_s(q)=k}} (y_q \in \mathcal{C}(k))$$

where

$$y_q = x_q \text{ for } q \neq i,$$
$$y_i = \partial(x_i).$$

One readily verifies that the sum d of all these maps $\widetilde{B}_{\mathcal{C}}(A) \to \widetilde{B}_{\mathcal{C}}(A)$ does, in fact, satisfy

$$d \circ d = 0.$$

This, in effect, follows from the anticommutation of the operations $\partial/\partial e_i$, and left multiplication by the generators f_j.

Proposition 7.

$$H(\widetilde{B}_{\mathcal{C}}(A), d) = 0.$$

Proof. Define, for a tree (n, s, L), a tree $(n+1, \overline{s}, \overline{L})$ by

$$\overline{L} = L \cup \{n+1\},$$
$$\overline{s}(i) = s(i) \text{ for } i < n,$$
$$\overline{s}(n) = n+1.$$

Define a map of degree 1

$$h : \widetilde{B}_{\mathcal{C}}(A) \to \widetilde{B}_{\mathcal{C}}(A)$$

by

$$\prod_i e_i \cdot \prod_j f_j \cdot \bigotimes_{j\in L}(x_j \in A) \otimes \bigotimes_{\substack{j \in \mathbf{n} \smallsetminus L \\ v_s(j)=k}} (x_j \in \mathcal{C}(k))$$

$$\mapsto e_n \cdot \prod_i e_i \cdot \prod_j f_j \cdot \bigotimes_{j\in \overline{L}}(y_j \in A) \otimes \bigotimes_{\substack{j \in (\mathbf{n+1}) \smallsetminus \overline{L} \\ v_{\overline{s}}(j)=k}} (y_j \in \mathcal{C}(k))$$

where

$$y_j = x_j \text{ for } j \in \mathbf{n},$$

$$y_{n+1} = 1.$$

(Note that the tree $(n+1, \overline{s}, \overline{L})$ is obtained from (n, s, L) by "grafting", i.e. by attaching a new root below the old root, and connecting them with an edge. The labels of the "grafter tree" stay the same, the label of the new root is 1.)

Then

$$dh + hd = Id.$$

□

Let

$$A \xrightarrow{\subseteq} \widetilde{B}_{\mathcal{C}}(A)$$

(3)

$$a \longmapsto a \in \bigoplus_{(1,*,\{1\})} A.$$

Denote

(4) $$B_{\mathcal{C}}(A) := (\widetilde{B}_{\mathcal{C}}(A)/A)[-1].$$

Then there is a map

(5) $$\mu : B_{\mathcal{C}}(A) \to A$$

given by

$$\bigotimes_{j \in L}(x_j \in A) \otimes \bigotimes_{\substack{j \in \mathbf{n} \smallsetminus L \\ v_s(j) = k}} (x_j \in \mathcal{C}(k))$$

$$\mapsto \theta(x_1, \ldots, x_{n-1}; x_n) \text{ if } (n, s, L) \text{ is a bush}$$

$$\mapsto 0 \text{ else.}$$

Proposition 8. *(1) The map (5) is a chain map, and an equivalence for an S-cofibrant DG K-module operad $\mathcal{C}$.*

(2) There is a natural DG $\mathcal{C}$-algebra structure on $B_{\mathcal{C}}(A)$ such that (5) is a map of DG $\mathcal{C}$-algebras.

Proof. For (1), to prove that (5) is a chain map, it suffices to prove that it vanishes on differentials of trees (n, s, L) where $\epsilon_{(n,s,L)}(n) = 2$. On such trees, however, the differential has two summands (one edge contraction and one leaf contraction), which cancel after applying (5) to them.

By definition, further, (5) fits into the following diagram of chain complexes:

$$(6)\qquad \begin{array}{ccccc} A[-1] & \longrightarrow & \widetilde{B}_{\mathcal{C}}(A)[-1] & \longrightarrow & B_{\mathcal{C}}(A) \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \widetilde{\mu}} & & \downarrow{\scriptstyle \mu} \\ A[-1] & \longrightarrow & A \otimes I[-1] & \longrightarrow & A \end{array}$$

where I is the chain complex of K-modules

$$K \xrightarrow{\cong} K$$

in dimensions $1, 0$.

The map $\widetilde{\mu}$ has

$$a \in \bigoplus_{(1,*,\{1\})} A \mapsto a.$$

Since the source and target of $\widetilde{\mu}$ are both acyclic, μ is an equivalence by (6) and the 5-lemma.

The $\mathcal{C}$-algebra structure on $B_{\mathcal{C}}(A)$ is given by

$$\prod_i e_i^q \prod_{j<n_q} f_j^q \cdot \prod_{q=1}^m f_{n_q}^q \cdot$$

$$\bigotimes_{q=1}^m \left(\bigotimes_{j\in L_q} (x_{j,q} \in A) \otimes \bigotimes_{\substack{j \in \mathbf{n_q} \smallsetminus L \\ v_s(j)=k}} (x_{j,q} \in \mathcal{C}(k)) \right) \otimes (x \in \mathcal{C}(m))$$

$$\mapsto \prod_q (\prod_i e_{\ell_q(i)} \cdot \prod_{j<n_q} f_{\ell_q(j)} \cdot f_n \cdot \bigotimes_{j\in L} (y_j \in A) \otimes \bigotimes_{\substack{j \in \mathbf{n} \smallsetminus L \\ v_s(j)=k}} (y_j \in \mathcal{C}(k))$$

where

$$\ell_q(i) = i + (n_1 - 1) + \cdots + (n_{q-1} - 1),$$

$$n = \sum_{q=1}^m n_q - m + 1,$$

for $s_q(j) \neq n_q$,

$$s(\ell_q(j)) = \ell_q(s_q(j)),$$

for $s_q(j) = n_q$,

$$s(\ell_q(j)) = n,$$

for $1 \leq j < n_q$,

$$y_{\ell_1(j)} = x_{j,q},$$

$$y_n = \gamma(x_{n,1}, \dots, x_{n,q}; x).$$

$\square$

5. D-structures

Proposition 9. *There exists a graded homomorphism of K-modules*

$$\bigoplus_{n\geq 0} \delta_n = \delta : \widetilde{B}_{\mathcal{C}}(A) \to \bigoplus_{n\geq 0} \widetilde{B}_{\mathcal{C}}(A)^{\otimes n} \otimes_{\Sigma_n} \mathcal{C}(n) = B_{\mathcal{C}}(A)$$

such that if we denote, for $x \in \widetilde{B}_{\mathcal{C}}(A)$,

$$\delta(x) = \sum (\delta'_n(x))_1 \otimes \cdots \otimes (\delta'_n(x))_n \otimes \delta''_n(x))$$

with $(\delta'_n(x))_i \in \widetilde{B}_{\mathcal{C}}(A)$, $\delta''_n(x) \in \mathcal{C}(n)$, then

$$d(x_1 \otimes \cdots \otimes x_n \otimes x_{n+1})$$

$$= \sum_{i=1}^{n+1} (-1)^{|x_1|+\ldots|x_{i-1}|} x_1 \otimes \cdots \otimes x_{i-1} \otimes dx_i \otimes x_{i+1} \otimes \cdots \otimes x_{n+1}$$

$$+ \sum_{i=1}^{n} \sum_{m\geq 0} (-1)^{|x_1|+\cdots+|x_i|+|\delta''_m(x_i)|\cdot(|x_{i+1}|+\cdots+|x_n|)} x_1 \otimes \ldots$$

$$\ldots x_i \otimes (\delta'_m(x_i))_1 \otimes \cdots \otimes (\delta'_m(x_i))_m \otimes x_{i+1} \otimes \ldots$$

$$\cdots \otimes x_n \otimes \gamma_i(\delta''_n(x_i), x_{n+1}).$$

Proof. For an element $x \in \widetilde{B}_{\mathcal{C}}(A)$ indexed over a tree of the form $(1, *, \{1\})$ (thus, $x \in A$), we put

$$\delta(x) = 0.$$

For elements indexed over a tree (n, s, L) with $n \notin L$ with successor trees (n_i, s_i, L_i), $i = 1, \dots, k$ (note that the case $n = 1$ may occur, in which case we have $k=0$), define

$$\delta\left(\prod_{\ell} e_\ell \cdot \prod_j f_j \cdot \bigotimes_{j\in L}(x_j \in A) \otimes \bigotimes_{\substack{j\in \mathbf{n}\smallsetminus L\\ v_s(j)=k}} (x_j \in \mathcal{C}(k))\right)$$

$$= \prod_{\ell} e_\ell \cdot \prod_j f_j \cdot \bigotimes_{i=1}^{k}\left(\bigotimes_{j=n_1+\cdots+n_{i-1}+1}^{n_1+\cdots+n_i} x_j\right) \otimes (x_n \in \mathcal{C}(k)).$$

Note carefully that this formula hides a sign coming from the shuffle of the e_ℓ's and f_j's so that the variables corresponding to the individual $\widetilde{B}_{\mathcal{C}}(A)$ factors on the right hand side are moved together. The sign in the second summand of the differential comes from the fact that the root of the trees corresponding to the i'th factor must be moved to the right to apply the operad operation. $\square$

Definition 10. Let $(\mathcal{C}, \partial)$ be a unital S-cofibrant DG-operad over a field K. A *D-structure* with respect to $\mathcal{C}$ is a DG-K-module (N, δ) together with a graded homomorphism of K-modules

$$\delta : N \to \bigoplus_{n\geq 0} N^{\otimes n} \otimes_{\Sigma_n} \mathcal{C}(n),$$

$$x \mapsto \sum_n ((\delta'_n(x))_1 \otimes \cdots \otimes (\delta'_n(x))_n \otimes \delta''_n(x)),$$

such that on

$$CN = \bigoplus_{n\geq} N^{\otimes n} \otimes_{\Sigma_n} \mathcal{C}(n),$$

the map $\Delta : CN \to CN$ given by

$$\begin{aligned}
&\Delta(x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) \\
&= \sum_{i=1}^{n+1} (-1)^{|x_1|+\dots|x_{i-1}|} x_1 \otimes \cdots \otimes x_{i-1} \otimes dx_i \otimes x_{i+1} \otimes \cdots \otimes x_{n+1} \\
&+ \sum_{i=1}^{n} \sum_{m\geq 0} (-1)^{|x_1|+\cdots+|x_i|+|\delta''_m(x_i)|\cdot(|x_{i+1}|+\cdots+|x_n|)} x_1 \otimes \dots \\
&\dots x_i \otimes (\delta'_m(x_i))_1 \otimes \cdots \otimes (\delta'_m(x_i))_m \otimes x_{i+1} \otimes \dots \\
&\cdots \otimes x_n \otimes \gamma_i(\delta''_n(x_i), x_{n+1}).
\end{aligned} \tag{7}$$

defines a differential on CN (i.e., $\Delta \circ \Delta = 0$), and furthermore, (CN, Δ), with the $\mathcal{C}$-algebra structure on CN in the category of graded K-modules coming from the monad, is a DG-$\mathcal{C}$-algebra.

Proposition 11. *$(\widetilde{B}_{\mathcal{C}}(A), \delta)$ is a $\mathcal{C}$-D-structure.*

Proof. This follows immediately from Proposition 9. □

Comment: Denote, for a $\mathcal{C}$-D-structure (N, Δ), by $C_\Delta(N)$ the DG-$\mathcal{C}$-algebra (CN, Δ). Note that the explicit notation is justified, as CN has its own $\mathcal{C}$-algebra structure coming from the fact that C is the monad defining $\mathcal{C}$-algebras. Recall also that the unit $\eta : N \to CN$ is a chain map.

The map $\eta : N \to C_\Delta(N)$, on the other hand, is not a chain map. In fact, it is easily seen that the definition implies

$$\eta d + \delta = \Delta \eta, \tag{8}$$

which in turn implies that

$$x \mapsto (-1)^{|x|} \delta(x)$$

is a chain map $\delta' : N \to C_\Delta(N)$. Further, this map is null-homotopic by (8). In the case of $N = \widetilde{B}_{\mathcal{C}}(A)$, δ' is the projection

$$\widetilde{B}_{\mathcal{C}}(A) \to B_{\mathcal{C}}(A)[1]$$

of (4). In view of Proposition 7, then, one may ask if every D-structure has 0 homology. We do not know if this is the case.

Definition 12. A *morphism of* $\mathcal{C}$*-D-structures* $f : (N, \Delta) \to (N', \Delta')$ is a homomorphism of graded K-modules

$$f_0 : N \to C_{\Delta'}(N')$$

which extends to a homomorphism of DG-$\mathcal{C}$-algebras $\overline{f} : C_\Delta(N) \to C_{\Delta'}(N')$ in the sense of the following diagram:

$$\begin{array}{ccc} N & \xrightarrow{f_0} & N' \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ C_\Delta(N) & \underset{\overline{f}}{\dashrightarrow} & C_{\Delta'}(N'). \end{array}$$

The *identity morphism* is η. For $g : (N', \Delta') \to (N'', \Delta'')$, composition is defined by

$$(g \circ f)_0 = \overline{g}\overline{f}\eta.$$

Comment: Note that $\overline{f}$ is uniquely determined as a map of $\mathcal{C}$-algebras because $C_\Delta(N)$ is equal to CN as a graded K-module and $\eta : N \to CN$ is a universal homomorphism of graded K-modules into $\mathcal{C}$-algebras in the category of graded K-modules. Thus, the condition on f_0 reduces to requiring that $\overline{f}$ be a chain map. This can be written explicitly as follows: For

$$x_1 \otimes \cdots \otimes x_n \otimes x_{n+1} \in N^{\otimes n} \otimes \mathcal{C}(n),$$

one requires that

$$\sum_{i=1}^{n}(-1)^{|x_1|+\dots|x_{i-1}|}f_0(x_1)\otimes\dots\otimes f_0(x_{i-1})\otimes df_0(x_i)\otimes f_0(x_{i+1})\otimes\dots\otimes f_0(x_n)\otimes x_{n+1}$$

$$+\sum_{i=1}^{n}\sum_{m\geq 0}(-1)^{|x_1|+\dots+|x_i|+|\delta''_m(x_i)|\cdot(|x_{i+1}|+\dots+|x_n|)}f_0(x_1)\otimes\dots$$

$$\dots f_0(x_i)\otimes(\delta'_m(f_0(x_i)))_1\otimes\dots\otimes(\delta'_m(f_0(x_i)))_m\otimes f_0(x_{i+1})\otimes\dots$$

$$\dots\otimes f_0(x_n)\otimes\gamma_i(\delta''_n(f_0(x_i)),x_{n+1})$$

$$=\sum_{i=1}^{n}(-1)^{|x_1|+\dots|x_{i-1}|}f_0(x_1)\otimes\dots\otimes f_0(x_{i-1})\otimes f_0(dx_i)\otimes f_0(x_{i+1})\otimes\dots\otimes f_0(x_n)\otimes x_{n+1}$$

$$+\sum_{i=1}^{n}\sum_{m\geq 0}(-1)^{|x_1|+\dots+|x_i|+|\delta''_m(x_i)|\cdot(|x_{i+1}|+\dots+|x_n|)}f_0(x_1)\otimes\dots$$

$$\dots f_0(x_i)\otimes f_0((\delta'_m(x_i))_1)\otimes\dots\otimes f_0((\delta'_m(x_i))_m)\otimes f_0(x_{i+1})\otimes\dots$$

$$\dots\otimes f_0(x_n)\otimes\gamma_i(\delta''_n(x_i),x_{n+1}).$$

Definition 13. A morphism $f:(N,\Delta)\to(N',\Delta')$ of $\mathcal{C}$-D-structures is an *equivalence* if the homomorphism $\overline{f}:C_\Delta(N)\to C_{\Delta'}(N')$ induces an isomorphism in homology.

Proposition 14. *Let $\mathcal{C}$ be an S-cofibrant DG K-module operad. The following functors preserve equivalences:*

$$\widetilde{B}_{\mathcal{C}}(?):\textit{DG-}\mathcal{C}\textit{-algebras}\to\mathcal{C}\textit{-D-structures}\tag{9}$$

$$C_\Delta(?):\mathcal{C}\textit{-D-structures}\to\textit{DG-}\mathcal{C}\textit{-algebras.}\tag{10}$$

There exist natural equivalences

$$C_\Delta(\widetilde{B}_{\mathcal{C}}(?))\to Id:\textit{DG-}\mathcal{C}\textit{-algebras}\to\textit{DG-}\mathcal{C}\textit{-algebras}\tag{11}$$

$$\widetilde{B}_{\mathcal{C}}(C_\Delta(?))\to Id:\mathcal{C}\textit{-D-structures}\to\mathcal{C}\textit{-D-structures.}\tag{12}$$

Proof. We have $C_\Delta\widetilde{B}_{\mathcal{C}}(A)=B_{\mathcal{C}}(A)$ by Proposition 9, so the natural equivalence of DG-$\mathcal{C}$-algebras (11) is established by Proposition 8. Since equivalences of $\mathcal{C}$-D-structures $f:(N,\Delta)\to(N',\Delta')$ are defined as equivalences of the corresponding DG-$\mathcal{C}$-algebras, this also implies that (9) preserves equivalences.

The functor (10) on morphisms is defined by $f \mapsto \overline{f}$. It preserves equivalences by definition.

To construct the natural equivalence ϵ of (12), again, we need to construct $\overline{\epsilon}$, i.e. a natural equivalence of DG-$\mathcal{C}$-algebras

$$B_{\mathcal{C}}(C_\Delta(?)) \to C_\Delta(?),$$

which is the natural equivalence of Proposition 8 applied to C_Δ. $\square$

6. Derived categories

The material covered in this section is well known. We cover it for the lack of convenient reference.

Definition 15. *Let C be a category, and let $\mathcal{E} \subseteq Mor(C)$ be an arbitrary class of morphism, which we call* equivalences. *A* derived category *DC is a category together with a functor*

$$\Phi : C \to DC$$

such that

$$\phi \in \mathcal{E} \Rightarrow \Phi(\phi) \text{ is an isomorphism,}$$

and for every functor $F : C \to B$ such that

$$\phi \in \mathcal{E} \Rightarrow F(\phi) \text{ is an isomorphism,}$$

there exists a functor $DF : DC \to D$ and a natural isomorphism

$$\eta : F \xrightarrow{\cong} DF \circ \Phi,$$

which is further unique in the following sense: For any other functor $DF' : DC \to D$ and a natural isomorphism

$$\eta' : F \xrightarrow{\cong} DF' \circ \Phi,$$

there exists a unique natural isomorphism

$$\xi : DF \xrightarrow{\cong} DF'$$

such that

$$\xi(\Phi) = \eta' \circ \eta^{-1}. \tag{13}$$

Note that the existence of a derived category is not automatic, and in particular cannot be proved by the usual algebraic "localization" argument because $\mathcal{E}$ is only a class, not necessarily a set. In certain cases a derived category is known to exist, for example when $\mathcal{E}$ is the class of equivalences in a Quillen model structure [4].

Proposition 16. *If a derived category exists then it is unique in the sense that for two functors $\Phi : C \to DC$, $\Phi' : C \to D'C$ both satisfying the condition of Definition 15, there exists a natural equivalence*

$$A : DC \to D'C,\ B : D'C \to DC, \tag{14}$$

$$\epsilon : Id \xrightarrow{\cong} BA, \quad \zeta : Id \xrightarrow{\cong} AB \tag{15}$$

and natural isomorphisms

$$\eta : \Phi' \xrightarrow{\cong} A\Phi, \quad \kappa : \Phi \xrightarrow{\cong} B \circ \Phi' \tag{16}$$

such that

$$(B\eta) \circ \kappa = \epsilon\Phi,\ (A\kappa) \circ \eta = \zeta\Phi'. \tag{17}$$

Furthermore, these data are unique in the following sense: For any other choice of the data (14), (15) satisfying (16), (17) (which we will decorate with a $(?)^\circ$ to make a distinction), there are unique isomorphisms

$$\alpha : A \xrightarrow{\cong} A^\circ, \quad \beta : B \xrightarrow{\cong} B \circ B^\circ$$

such that

$$Id = (\zeta^\circ)^{-1} \circ (\beta A^\circ) \circ (B\alpha) \circ \zeta,$$
$$Id = (\epsilon^\circ)^{-1} \circ (\alpha B^\circ) \circ (A\beta) \circ \epsilon,$$
$$\eta^\circ = \alpha\Phi \circ \eta,\ \kappa^\circ = \beta\Phi' \circ \kappa.$$

Proof. One obtains the transformations η, κ as special cases of the η in Definition 15, and the $\epsilon, \zeta, \alpha, \beta$ as special cases of the ξ of Definition 15. The diagrams are then special cases of (13) and the uniqueness of ξ in Definition 15. □

Lemma 17. *Suppose C_1, C_2 are categories with classes of equivalences $\mathcal{E}_1$, $\mathcal{E}_2$. Suppose that*

(1) $(C_1, \mathcal{E}_1)$ has a derived category $\Phi_1 : C_1 \to DC_1$.

(2) There exists a pair of functors $F : C_1 \to C_2$, $G : C_2 \to C_1$ which preserve equivalences, and natural equivalences

$$\epsilon : FG \xrightarrow{\sim} Id, \quad \zeta : GF \xrightarrow{\sim} Id.$$

Then C_2 has a derived category $\Phi_2 : C_2 \to DC_2$, and there exist functors

$$DF : DC_1 \to DC_2,\ DG : DC_2 \to DC_1$$

and natural isomorphisms

$$D\epsilon : DFDG \xrightarrow{\cong} Id, \quad D\zeta : DGDF \xrightarrow{\cong} Id,$$

$$\kappa : \Phi_2 F \xrightarrow{\cong} DF\Phi, \quad \lambda : \Phi_1 G \xrightarrow{\cong} DG\Phi_2$$

satisfying the identities

$$Id = D\zeta \circ (DG\kappa) \circ (\lambda F) \circ \epsilon^{-1},$$
$$Id = D\epsilon \circ (DF\lambda) \circ (\kappa F) \circ \zeta^{-1}.$$

Comment: We say that F, G *induce an equivalence of derived categories.* There is also a uniqueness statement which we omit.

Proof. Put $DC_2 := DC_1$, $\Phi_2 := \Phi_1 G$. We may then define $DF = DG := Id$, and $\lambda := Id$. To define $\kappa : \Phi_1 GF \to \Phi_1$, put $\kappa := \Phi_1\zeta$. The naturel isomorphisms $D\zeta$, $D\epsilon$ are then defined as special cases of the ξ of Definition 15. □

Comment: The fact that we defined $DC_2 := DC_1$ does not affect the force of this Lemma, since a derived category is only defined up to equivalence; the force of the statement is in the functors and natural isomorphisms involved.

Theorem 18. *Let $\mathcal{C}$ be an S-cofibrant DG K-module operad. There exists a derived category of $\mathcal{C}$-D-structures, and the functors of Proposition 14 induce an equivalence of derived categories.*

Proof. An immediate consequence of Proposition 14 and Lemma 17. □

References

[1] M.Anel, A.Joyal: Sweedler Theoy for (co)algebras and the bar-cobar constructions, arXiv:1309.6952

[2] J.M.Boardman, R.M.Vogt: *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973

[3] W.G.Dwyer, D.M.Kan: Homotopy theory and simplicial groupoids, *Nederl. Akad. Wetensch. Indag. Math.*, 46, (1984), 379 - 385

[4] W.Dwyer, J. Spalinski: Homotopy theories and model categories, *Handbook of algebraic topology*, 73126, North-Holland, Amsterdam, 1995

[5] V.Ginzburg, M.Kapranov: Koszul duality for operads, *Duke Math. J.* 76 (1994) 203-272

[6] W.Gan: Koszul Duality for Dioperads, *Math. Res. Letters* 10 (2003) 109-124

[7] E.Getzler, J.D.S.Jones: A_∞-algebras and the cyclic bar complex, *Illinois J. Math.* 34 (1990), no. 2, 256-283

[8] J.Hirsh, J.Milles: Curved Koszul duality theory, *Math. Ann.* 354 (2012), 1465-1520

[9] P.Hu: Higher string topology on general spaces, *Proc. London Math. Soc.* (3) 93 (2006), no. 2, 515-544

[10] P.Hu, I.Kriz, A.A.Voronov: On Kontsevich's Hochschild cohomology conjecture. (English summary) *Compos. Math.* 142 (2006), no. 1, 143-168

[11] M.Kontsevich: Deformations of algebras over operads and the Deligne conjecture, *Conférence Moshé Flato 1999*, Vol. I (Dijon), 255-307, Math. Phys. Stud., 21, Kluwer Acad. Publ., Dordrecht, 2000

[12] M.Kontsevich, Y.Soibelman: Deformations of algebras over operads and the Deligne conjecture, *Conférence Moshé Flato 1999*, Vol. I (Dijon), 255-307, Math. Phys. Stud., 21, Kluwer Acad. Publ., Dordrecht, 2000

[13] I.Kriz, J.P.May:*Operads, algebras, modules and motives*, Astrisque No. 233 (1995)

[14] R.Lipshitz, P.S.Ozsváth, D.P.Thurston: Bimodules in bordered Heegaard Floer homology, arXiv: 1003.0598

[15] J.-L. Loday, B. Vallette: *Algebraic Operads*, Grundlehren der mathematischen Wissenschaften, Springer Verlag, 2012

[16] J.P. May: *The geometry of iterated loop spaces*, Lectures Notes in Mathematics, Vol. 271. Springer-Verlag, Berlin-New York, 1972

[17] J.P.May, R.Thomason: The uniqueness of infinite loop space machines, *Topology* 17 (1978), no. 3, 205224

[18] P.Ozsváth, Z.Szabó: Holomorphic disks and topological invariants for closed three-manifolds, *Ann. of Math.* (2) 159 (2004), no. 3, 10271158

[19] A.Polishchuk, L.Positelski: *Quadratic algebras*, Univ. Lecture Series 37, AMS, Providence, 2005

[20] S. Priddy: Koszul resolutions, *Trans. Amer. Math. Soc.* 152 1970 39-60

[21] D. Quillen: On the (co)homology of commutative rings, Homotopy theories and model categories. Handbook of algebraic topology, 73126, North-Holland, Amsterdam, 1995. *1970 Applications of Categorical Algebra* (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968) pp. 6587. Amer. Math. Soc., Providence, R.I.

[22] B.Vallette: A Koszul duality for PROPs, *Trans. Amer. Math.Soc.* 152 (1970) 39-60