A UNIVERSAL APPROACH TO VERTEX ALGEBRAS

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1. Introduction

The notion of vertex algebra due to Borcherds [2] and Frenkel-Lepowsky-Meurman [7] is one of the fundamental concepts of modern mathematics. The details of the theory are often considered difficult. This has been largely explained by the complexity of the core of the definition of the concept, a rather complicated type of Jacobi identity ([7]). But even in the recent simplification where the identity is replaced by the notion of locality ([4], cf. [6, 11], see also (20) below), the computations in proofs can be non-transparent, and in particular the constructions involved in examples are often contraintuitive for non-experts.

Deeper reasons for the difficulties may lie in the motivation, indeed the origins, of the concept of vertex algebra in quantum field theory. Thus, besides advanced algebra and complex analysis one might also need ideas from physics to gain the right intuition. Yet, the vertex algebra can be ultimately viewed and studied as a concept of pure algebra. In this vein, efforts of "algebraic" conceptualizations have been made. Notably there is the beautiful definition by Borcherds [3] which substantially generalizes vertex algebra (covering also some other similar models for higher dimensional quantum field theory) based on the notion of vertex group and relaxed multilinear category (see also Soibelman [17]), and the concept of chiral algebras by Beilinson and Drinfeld [1], which naturally incorporates "sheafification" on algebraic varieties. And, very important for our approach, there is the work of Roitman who in a series of papers [14, 15, 16] explored the notion of a free vertex algebra restricted by specific bounds of locality between given generators.

In this paper we approach vector algebras from the point of view of universal algebra. Such a perspective serves not so much to solving structural questions as, rather, a framework for constructions. In an analogy with group theory: the idea of free group, and presenting general groups in terms of generators and defining relations was not primarily intended as a tool for investigating the structure of groups, however useful it someties proved for that, too; shifting the focus from examples of groups naturally appearing in nature to investigating the properties of models one can write down changed the outlook on the field.

Not only in groups but generally, in standard categories of universal algebra (i.e., structures defined in terms of operations and relations, preferably equations between them) one can construct general objects from generators and defining relations. Now while the general vertex algebra is not an object of universal algebra suitable for such treatment, the concept of the k-connective vertex algebra (that

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is, a vertex algebra with $V_n = 0$ for $n < k$) are of that type, and constructions by generators and defining relations become formally possible.

The above mentioned papers of Roitman contain very interesting results such as free field realizations of his free objects, as well as highly curious examples of vertex algebras containing a given commutative non-associative algebra with invariant symmetric non-degenerate bilinear form in the most recent one [16]. In our approach we do not impose an explicit bound on the locality of any elements; instead, a lower bound on dimension is assumed (and in a certain weakened sense we can avoid the bounds altogether).

We make use of an older concept of an *operad* originally invented by J.P.May [13] in algebraic topology as an approach to infinite loop space theory. More precisely, we use a certain algebraic counterpart of the notion (see Ginzburg-Kapranov [8]) as a tool for describing universal algebras which have an underlying structure of a vector space, and n-linear operations, and suitable kinds of relations. Our main aim is in describing vertex algebras by means of (algebraic) operads and thus gaining at least some explicit insight into the structure of free objects.

A certain drawback of the Roitma's approach that we wish to avoid is the special role played by the generatora on which the locality bound is imposed. As such, the category of the objects thus obtained does not have the desired universal algebraic properties; in particular, it cannot be described as the category of algebras over an operad. This is parallelled in the "physical" properties of Roitman's free vertex algebras: although those objects are very nicely behaved in the sense that they can be embedded into lattice vertex algebras, in the case when the bounds are positive, (which, in our context, occurs for generators of weight > 1), they are not connective, in fact not bounded below at all. Accordingly, the lattice vertex algebras into which Roitman embeds his free objects are in these cases not associated with positive-definite lattices (which is, perhaps, not surprising, as also in physics, "free field", or Feigin-Fuchs realizations of interesting vertex algebras usually involve indefinite lattices - for one of many such references, see [10]). Among our algebras, the Roitman algebras occur as those with connectivity constraints additionally imposed. Therefore, our presentations in generators and defining relations actually coincide with Roitman's when they satisfy connectivity constraints (this happens, for example, for the case of positive-definite lattices, see Section 4 below).

Without assuming connectivity, it turns out that the right notion is actually a co-operad (a structure dual to operad) which can be described explicitly as the "space of all possible functions which can occur as correlation functions": we call this the *correlation function co-operad.* Vertex algebras then are precisely the same thing as algebras (in a suitable sense) over the correlation function co-operad (Theorem 13 below). Moreover, because of a certain finiteness property, k-connective vertex algebras can actually be precisely described as algebras over a suitable operad, whose structure we can determine quite explicitly (Theorem 10 below). Because of general theory of algebras over operads, we therefore have fairly precise control over free objects in the category of k-connective vertex algebras, and also over constructions by generators and defining relations (we give some examples). This is the main contribution of the present paper. Since it seemed hard to find a good comprehensive reference detailing the categorical properties of algebras over operads and co-operads, we also include a sizeable appendix containing the relevant categorical and universal algebra setup.

The paper is organized as follows: In Section 2, we describe the co-operad of correlation functions and its k-connective version. In Section 3, we prove our main results characterizing vertex algebras and k-connective vertex algebras in this setting. Section 4 contains examples of presentations of vertex algebras in terms of generators and defining relations. Section 5 is the Appendix as mentioned above.

2. VA-algebras and the correlation function co-operad

In this paper, all vector spaces we consider will be over the field of complex numbers $\mathbb{C}$. We shall say that a meromorphic function $f(z)$ on $\mathbb{C}P^1$ (the one-dimensional complex projective space) is *non-singular at* ∞ when the limit of $f(z)$ at ∞ is 0.

Definition 1. A *local function* is a meromorphic n-variable function on $\mathbb{C}P^1$ which is non-singular when all the variables are different and different from ∞.

The space of all local functions in variables $z_1,...z_n$ will be denoted by $\mathcal{C}(z_1,...,z_n)$.

Lemma 2. *There exists a vector space basis* B_n *of* $\mathcal{C}(z_1,...,z_n)$ *such that*

$$B_0 = \{1\}, \tag{1}$$

$$B_{n+1} = \{(z_{n+1} - z_i)^k,\ z_{n+1}^\ell \mid i = 1,...,n, 0 > k \in \mathbb{Z}\ 0 \leq \ell \in \mathbb{Z}\} \cdot B_n. \tag{2}$$

Proof: Let $f = f(z_1,...,z_{n+1}) \in \mathcal{C}(z_1,...,z_{n+1})$. Then consider f as a function of z_{n+1} with $z_1,...,z_n$ constant. Since every holomorphic function on $\mathbb{C}P^1$ is constant, there exist unique functions $g_k\ell$, g_{1k},...,g_{nk}, $0 > k \in \mathbb{Z}$, $0 \leq \ell \in \mathbb{Z}$, all but finitely many of which are 0, such that

$$f = \sum_k g_\ell z_{n+1}^\ell + \sum_{i,k} g_{ik}(z_{n+1} - z_i)^k. \tag{3}$$

Clearly, $g_\ell,\ g_ik \in \mathbb{C}(z_1,...,z_n)$, so an induction completes the proof. □

Clearly, there is a right action of Σ_n on $\mathcal{C}(z_1,...,z_n)$ given, for a permutation $\sigma : \{1,..,n\} \to \{1,...,n\}$, by

$$f\sigma(z_1,...,z_n) = f(z_{\sigma(1)},...,z_{\sigma(n)}). \tag{4}$$

We shall also denote $\mathcal{C}(z_1,...,z_n)$ by $\mathcal{C}(n)$, $n \geq 0$.

Now note that it follows from Lemma 2 that each of the vector spaces $\mathcal{C}(n)$ is graded by *the negative of the degree* (a function $f(z_1,...,z_n)$ is homogeneous of degree $k \in \mathbb{Z}$ if $f(az_1,..,az_n) = a^k f(z_1,...,z_n)$. Denote by $\mathcal{C}(n)_k$ the subspace of elements of degree k. (The sign reversal will be needed when relating our concept to the concept of vertex algebras; in writing fields of vertex algebras, the degree of a map attached as a coefficient at z^n *minus* n is a constant.)

For $k = p + q$, we shall now describe a map

$$\Phi_{n,m} : \mathcal{C}(n)_k \to \mathcal{C}(m+1)_p \otimes \mathcal{C}(n-m)_q. \tag{5}$$

This is done as follows: consider a function $f \in \mathcal{C}(z_1, ..., z_n)_k$. By Lemma 2, f can be expressed (non-uniquely) as a polynomial in

$$(z_i - z_j)^{-1}, \tag{6}$$

$$z_i. \tag{7}$$

Now modify this polynomial by replacing every monomial (6) with $i > m$, $j \leq m$ by

$$((t - z_j) + (z_i - t))^{-1}, \tag{8}$$

expanded in increasing powers of $z_i - t$, and every (7) with $i > m$ by

$$(t + (z_i - t)). \tag{9}$$

Additionally, now substitute t_s for $z_{s+m} - t$ with $s = 1, ..., n - m$ and z_{m+1} for t. In particular, a term (6) will remain unchanged for $i, j \leq m$, while for $i, j > m$, the term (6) will turn into

$$(t_{i-m} - t_{j-m})^{-1}.$$

Picking out terms of degree p in the variables $z_1, ..., z_{m+1}$, and degree q in the variables $t_1, ..., t_{n-m}$, we obtain an element

$$g \in \mathcal{C}(z_1, ..., z_{m+1})_p \otimes \mathcal{C}(t_1, ..., t_{n-m})_q \tag{10}$$

(one easily sees that the number of summands of the given fixed degree is finite). But clearly, (10) is a rewrite of (5).

Lemma 3. *The element g of (10) does not depend on the representation of f as a polynomial in (6), (7).*

Proof: Interpret $z_1, ..., z_m$ as constants different from each other and different from ∞. Pick $t \in \mathbb{C}P^1$ different from $z_1, ..., z_m$ and from ∞. Then, putting

$$P = \prod_{i \neq j} (z_i - z_j)^N$$

for a sufficiently large integer N, gP can be characterized as the Taylor expansion of fP as a function in $z_{m+1}, ..., z_n$ in the neighborhood of

$$(z_{m+1}, ..., z_n) = (t, ..., t).$$

$\square$

From now on, we will be using notation and certain facts from category theory and the theory of operads. These facts are somewhat technical and independent of the rest of the material, and are treated in the Appendix.

Theorem 4. *There exists a unique graded co-operad structure on the sequence $(\mathcal{C}(n))$ (we call this the* correlation function co-operad*) such that Σ_n-equivariance is given by (4) and the co-composition operation corresponding to inserting $n - m$ variables to the $m + 1$'st variable among variables indexed $1, ..., m, m + 1$ is given by $\Phi_{n,m}$.*

Proof: First of all, our the 'insertion' co-operations we defined are less general than those of the form (81). A general co-operation (81) can be obtained by repeated application of insertions together with equivariance.

The axioms we then need to verify are the version of equivariance for insertions, commutativity of insertions into two different original variables, and associativity of insertions (arising when an insertion is followed by another insertion into the new variables).

Regarding proving these properties, first note that equivariance is obvious, expressing simply symmetry of the construction in the variables involved in equal capacity. Regarding commutativity of insertions, using (8), it corresponds to the fact that expanding

$$((s-t)+(z_i-s)-(z_j-t))^{-1}$$

in increasing powers of (z_i-s) and then increasing powers of (z_j-t) gives the same series as expanding in (z_j-t) first and in (z_i-s) second. One readily verifies that the result in both cases is

$$\sum_{m,n\geq 0}\binom{m+n}{n}(-1)^n(z_j-s)^n(z_i-t)^m(s-t)^{-m-n-1}.$$

Coassociativity is equivalent to saying that expanding

$$(-(z_i-s)+(z_j-s))^{-1}$$

in increasing powers of the second summand, and then write (z_j-s) as

$$((t-s)+(z_j-t))$$

and expand in increasing powers of the second summand gives the same result as expanding

$$(-(z_i-t)+(z_j-t))^{-1}$$

in the second summand and then writing $-(z_i-t)$ as

$$(-(z_i-s)+(t-s))$$

and expanding in increasing powers of the second summand. Computation shows that both computation yield

$$-\sum_{m,n\geq 0}\binom{m+n}{m}(z_j-t)^n(t-s)^m(z_i-s)^{-n-m-1}.$$

□

Definition 5. A VA-algebra is a graded algebra over the graded co-operad $\mathcal{C}$.

We shall also be interested in cutting off VA-algebras by connectivity.

Definition 6. A *bounded VA-algebra* is a VA-algebra X such that for every $x_1,...,x_n\in X$, the image of

$$x_1\otimes ...\otimes x_n$$

under the coalgebra structure map in

$$X\otimes \mathcal{C}(n)_k$$

is 0 for $k<-N$ where N depends only on $x_1,...,x_n$.

Now let $k \in \mathbb{Z}$. Consider $\mathcal{C}$ as a $\mathbb{Z}$-sorted co-operad in the obvious way. Note that then in particular,

$$\mathcal{C}(n, n_1, ..., n_m) \text{ only depends on } n_1 + ... + n_m - n \tag{11}$$

(the difference $n_1 + ... + n_m - n$ being the degree).

Now consider the $\coprod_{n\geq 0} \mathbb{Z}^{n+1}$-sorted vector subspace $\mathcal{C}'_k$ of $\mathcal{C}$ which in degree $(n, n_1, ..., n_m)$ is equal to $\mathcal{C}(n, n_1, ..., n_m)$ when $n \geq k$, and 0 when $n < k$. By Proposition 29, Lemma 33 and Remark 34, there is a universal largest $\mathbb{Z}$-sorted sub-co-operad $\mathcal{C}_k \subset \mathcal{C}$ contained in $\mathcal{C}'_k$.

Definition 7. A k-connective VA-algebra is an algebra over the co-operad $\mathcal{C}_k$.

Lemma 8. *For every $k \in \mathbb{Z}$, and every $n, n_1, ..., n_m$, the vector space $\mathcal{C}_k(n, n_1, ..., n_m)$ is finite-dimensional.*

Proof: Suppose the dimension of $\mathcal{C}_k(n, n_1, ..., n_m)$ is infinite. Then the degree of singularity of elements at $z_i \to z_j$ is unbounded for some $i \neq j$ (in fact, we may choose $i = 1$, $j = 2$ by symmetry). Therefore, by Lemma 2, we must have elements which map by the comultiplication into non-zero elements of

$$\mathcal{C}(n, i, n_3, ..., n_m) \otimes \mathcal{C}(i, n_1, n_2) \tag{12}$$

with i arbitrarily low. But for $i < k$,

$$\mathcal{C}_k(n, i, n_3, ..., n_m) \otimes \mathcal{C}_k(i, n_1, n_2) = 0 \tag{13}$$

(since the second factor is 0), which is a contradiction. □

By Lemma 8, and Remark 32, the category of k-connective VA-algebras is the category of algebras over and operad, and in particular has free objects and factorization (see Remark 34).

3. The main results

We begin with a more precise statement than Lemma 8. Let $1 \leq i_1 < ... < i_p \leq m$. Let us introduce an increasing filtration on $\mathcal{C}(z_1, ..., z_m)$ as follows: For $N < 0$, let $F(i_1, ..., i_p)_N \mathcal{C}(z_1, ..., z_m) = 0$. For $N \geq 0$, let

$$\begin{array}{l} F(i_1, ..., i_p)_N \mathcal{C}(z_1, ..., z_n) := \langle f \in \mathcal{C}(z_1, ..., z_n) \mid \\ \text{Fixing a set } S \text{ of points } z_j \in \mathbb{C},\ j \neq i_1, ..., i_p, \text{ and} \\ \text{assuming they are all different, } \prod (z_{i_q} - z_{i_s})^{k_{qs}} f \text{ is} \\ \text{holomorphic on } (\mathbb{C}-S)^p \text{ for some integers } k_{qs} \geq 0, \\ \sum k_{ij} \leq N \rangle. \end{array} \tag{14}$$

Lemma 9. *For $N \geq 0$, there exists a vector space basis $B(i_1, ..., i_p)^N_n$ of*

$$F(i_1, ..., i_p)_N \mathcal{C}(z_1, ..., z_n)$$

such that

$$B(i_1, ..., i_p)^N_0 = \{1\}, \tag{15}$$

$$\begin{array}{l} B(i_1, ..., i_p)^N_{n+1} = \{(z_{n+1} - z_i)^k,\ z^\ell_{n+1} \mid \\ i = 1, ..., n, 0 > k \in \mathbb{Z}\ 0 \leq \ell \in \mathbb{Z}\} \cdot B(i_1, ..., i_p)^N_n. \end{array} \tag{16}$$

when $n+1 \neq i_p$*, and*

$$B(i_1,...,i_p)^N_{n+1} = \tag{17}$$
$$\bigcup_{k+M\leq N} \{(z_{n+1}-z_{i_s})^k \mid s=1,...,p-1, 0>k\in\mathbb{Z}\,\}\cdot B(i_1,...,i_{p-1})^M_n.$$
$$\cup\{(z_{n+1}-z_i)^k, z^\ell_{n+1},\ 0>k, i\neq i_1,...,i_{p-1}, 0\leq \ell\in\mathbb{Z}\}\cdot B(i_1,...,i_{p-1})^N_n$$

when $i_p = n+1$*.*

Theorem 10. *If* $k<0$*, then* $\mathcal{C}_k(n,n_1,...,n_m))=0$ *for all* $n,n_1,...,n_m$*. In general,* $\mathcal{C}_k(n,n_1,...,n_m)\subseteq\mathcal{C}(z_1,...,z_m)$ *is the subspace of all* f *such that for any* $1\leq i_1< ...<i_p\leq m$*,*

$$f\in F(i_1,...,i_p)_N\mathcal{C}(z_1,...,z_m) \tag{18}$$

where

$$N=-k+n_{i_1}+...+n_{i_p}. \tag{19}$$

Proof: The statement for $k<0$ immediately follows from the general statement by considering $p=0$. In the general case, our insertion operation clearly implies that $\mathcal{C}_k(n,n_1,...,n_m)$ is contained in the vector subspace suggested. On the other hand, since the insertion operation does not decrease the powers of the differences of the inserted variables involved (i.e. "does not increase singularity in the inserted variables"), the suggested subspaces form a $\mathbb{Z}$-sorted sub-co-operad. □

Because of variations in the definition, let us define our notion of vertex algebra.

Definition 11. A *vertex algebra* is a Z-graded vector space V together with, for each $a\in V_k$, a series

$$a(z)=\sum_n a(n)z^{-n-1} \tag{20}$$

where

$$a(n):V_m\rightarrow V_{m-n+k-1},$$

an operator $L_{-1}:V_n\rightarrow V_{n+1}$, and an element $1\in V_0$ such that the following properties hold:

(21) For every $a,b,c\in V$ and every $m\in\mathbb{Z}$, there exists an $N\in\mathbb{N}$ such that the $V_m[[z,t]]$-summand $s_m(z,t)$ of $(a(z)b(t)-b(t)a(z))c$ satisfies $(z-t)^N s_m(z,t)=0$.

$$[L_{-1},a(z)]=\partial_z a(z) \tag{22}$$

(the symbol ∂_z denotes (partial) derivative by z),

$$L_{-1}1=0, \tag{23}$$

$$1(z)=1, \tag{24}$$

$$a(n)1=0 \text{ for } n\geq 0. \tag{25}$$

We will call a vertex algebra uniform *when the axiom (21) is replaced by the following obviously stronger statement:*

(26) *For $a, b \in V$, $a(z)$ and $b(t)$ are local in the sense that $(z-t)^N(a(z)b(t) - b(t)a(z)) = 0$ for some integer $N \geq 0$ (possibly dependent on a, b).*

Comment: The axioms (21), (26) are known as locality axioms. Traditionally (cf. [11]), the stronger axiom (26) are used. In this paper, we consider $\mathbb{Z}$-*graded* vertex algebras. In this context, the axiom (26) has the non-uniform version (21). On the other hand, in [11], one considers vertex algebras without a $\mathbb{Z}$-grading, in which case the non-uniform axiom does not appear to make sense. One has the following result:

Lemma 12. [14] *A uniform vertex algebra satisfies the following property:*

(27) *For any $a, b \in V$ there exists $K \in \mathbb{Z}$ such that $a(i)b = 0$ for $i > K$.*

Proof: By Kac [11], a uniform vertex algebra satisfies the identity

$$a(m)b(k) - b(k)a(m) = \sum_{j \geq 0} \binom{m}{j} (a(j)b)(m+k-j). \tag{28}$$

When (26) holds with a uniform N, then we claim that (27) must hold with $K = N$. Indeed, if $a(i)b \neq 0$ for $i > N$, then for $k + m - i = -1$,

$$a(z)b(t) - b(t)a(z)$$

applied to 1 has the summand

$$\sum_{m \in \mathbb{Z}} \binom{m}{i} a(i)bz^m t^{-1-m+i},$$

which is not annihilated by $(z-t)^N$. □

Theorem 13. *There is a canonical equivalence between the category of vertex algebras (resp. uniform vertex algebras, resp. vertex algebras V with $V_n = 0$ for $n < k$) and VA-algebras (resp. bounded VA-algebras resp. k-connective VA-algebras). Moreover, the equivalence commutes with the forgetful functors to $\mathbb{Z}$-graded vector spaces.*

Proof: First, assume V is a vertex algebra according to the Definition 11. Then generalizing the usual arguments (cf. [11, 6]), one can show that V possesses correlation functions. In one way, an $n+2$-point correlation function can be described, for any integers $k_0, k_1, ..., k_n, k_\infty$ as a linear combination

$$\phi(a_0, a_1, ..., a_n)(0, z_1, ..., z_n) = \sum b_j f_j(0, z_1, ..., z_n) \tag{29}$$

with coefficients $b_j \in V_{k_\infty}$ of local functions (in the sense of definition 1) in variables z_0, z_1,...,z_n, dependent n-linearly on homogeneous elements $a_i \in V_{k_i}$, $i = 0, ..., n$, such that ϕ has expansion

$$a_n(z_n)...a_1(z_1)(a_0) \tag{30}$$

convergent in the range

$$|z_n| > ... > |z_1| > 0.$$

Additionally, one can choose

$$f(z_0, z_1, ..., z_n) = \exp(-z_0 L_{-1}) f(0, z_1 - z_0, ..., z_n - z_0). \tag{31}$$

The structure map

$$V_{k_0} \otimes V_{k_1} \otimes ... \otimes V_{k_n} \to \mathcal{C}(z_0, z_1, ..., z_n)_{k_\infty - k_0 - ... - k_n} \otimes V_{k_\infty} \tag{32}$$

is then defined by

$$a_0 \otimes a_1 \otimes ... \otimes a_n \mapsto \phi(a_0, a_1, ..., a_n)(z_0, z_1, ..., z_n) := \sum b_j f_j(z_0, ..., z_n). \tag{33}$$

To show that this defines a co-operad algebra structure, equivariance is a well known property (see e.g. Kac [11]). To prove compatibility with the insertion operator, one can use the well known fact that

$$\phi(a_0, a_1, ..., a_n)(z_0, z_1, ..., z_n)$$

also has an expansion

$$(a_n(z_n - t)...a_{m+1}(z_{m+1} - t)a_m)(t)a_{m-1}(z_{m-1})...a_1(z_1)a_0(z_0)1$$

convergent for

$$\begin{aligned} &1 > |t| > |z_{m-1}| > ... > |z_1| > |z_0|, \\ &|t| - |z_{m-1}| > |z_n - t| > ... > |z_{m+1} - t| > 0. \end{aligned}$$

In the uniform case, to prove boundedness, use Lemma 2, Lemma 12 and axiom (27) repeatedly.

On the other hand, assuming V has the structure of a graded $\mathcal{C}$-algebra, and letting, under (32), for $n = 1$, as above,

$$a_0 \otimes a_1 \mapsto \phi(a_0, a_1)(z_0, z_1), \tag{34}$$

we let the degree k_∞ summand of

$$a_1(z)(a_0)$$

be the Taylor expansion at 0 of

$$\phi(a_0, a_1)(0, z).$$

Let also, as usual, $L_{-1}a$ be defined as the constant term of

$$(a(z)1)'.$$

To prove locality, first note that by the fact that a meromorphic function on $\mathbb{C}P^1$ is the sum of the singular parts of its expansions at all points (which, in turn, is due to the fact that a non-singular function on all of $\mathbb{C}P^1$ is 0), the co-composition axiom and the identity

$$z^m = \sum_{i \geq 0} \binom{m}{i} (z - t)^i t^{m-i} \text{ for } |z - t| < |t| \tag{35}$$

imply that

$$z^m \phi(?, a, b)(0, z, t) \tag{36}$$

is anywhere on $\mathbb{C}P^1$ equal to

$$\begin{aligned}(37)\qquad &\sum_{p\geq m} b(t)a(p)z^{m-p-1}+\sum_{p<m} a(p)b(t)z^{m-p-1}\\ &+\sum_{0\leq i\leq j}\binom{m}{i}(z-t)^{i-j-1}(a(j)b)(t)t^{m-i}.\end{aligned}$$

Subtracting z times (36) from the analogous expression obtained by replacing m by $m+1$, we obtain

$$(38)\qquad a(m)b(t)-b(t)a(m)$$

on the left hand side and

$$\begin{aligned}(39)\qquad &\sum_{0\leq i\leq j}\binom{m+1}{i}(z-t)^{i-j-1}(a(j)b)(t)t^{m+1-i}\\ &-((z-t)+t)\sum_{0\leq i\leq j}(z-t)^{i-j-1}(a(j)b)(t)t^{m-i}.\end{aligned}$$

Using

$$\binom{m+1}{i}=\binom{m}{i}+\binom{m}{i-1}\text{ for } i>0,$$

after chain cancellation, the only surviving term of (39) is the term corresponding to $\binom{m}{i-1}$ and $i=j+1$ in the first summand , i.e.

$$(40)\qquad \sum_{0\leq i}\binom{m}{j}(a(j)b)(t)t^{m-j}$$

Equating (38) and (40) and taking the coefficient at t^{-k-1} gives (28), which is a part of the Jacobi identity. Even more importantly for us, however, the graded co-operad algebra structure implies that when applied to an element c, the summand of given degree m on the right hand side of (28) has j bounded above, say, by N. Grouping terms of (28) with $m+k$ constant, we then see that the V-degree m summand of

$$(41)\qquad [a(z),b(t)]c$$

is equal to a sum of terms which are V-degree m summands of some constant times a power of z times $(a(i)b)(t)c$ times the i'th derivative of the delta function $\delta(z/t)$ (say, by z), for $0\leq i\leq N$. Thus, (41) is annihilated when multiplied by $(z-t)^N$, as required.

One also sees clearly that in the bounded case, the terms of (28) with $j>N$ for a constant N dependent on a,b vanish outright, which implies uniformity.

To prove the axiom (22), consider the following diagram from the definition of algebra over a co-operad:

$$(42)\qquad \begin{CD} X\otimes X @>{\theta}>> X\otimes\mathcal{C}(2)\\ @V{\theta}VV @VV{\theta\otimes 1}V\\ X\otimes\mathcal{C}(2) @>>{1\otimes\gamma}> X\otimes\mathcal{C}(1)\otimes\mathcal{C}(2). \end{CD}$$

Our plan is to consider the image of $a\otimes b$ under (42); we will write the variable corresponding to a (resp. b) under θ as z (resp. t). We then perform insertion to the variable t, calling the new variable u, and we shall set $t = 0$, and take the linear term with respect to u. Taking the path through the upper right corner of (42), we obtain

$$aL_{-1}(z)b, \tag{43}$$

while taking the path through the lower left corner, we obtain two terms corresponding to u-linear terms of the expansion

$$z^n = \sum_{i\geq 0} \binom{n}{i} (z-u)^{n-i} u^i,$$

namely coming from $-n(z-u)^{n-1}$ and $(z-u)^n u$. In this order, this corresponds to the first and second term of the sum

$$-a'(z)b + L_{-1}a(z)b. \tag{44}$$

The equality between (43) and (44) is (22). All the other axioms are obvious, which completes the proof of the statement of the Theorem for general and bounded VA-algebras.

The statement for the k-connective follows easily: clearly, if we have a k-connective VA algebra X, then by applying

$$\theta : X \to X \otimes \mathcal{C}(1)$$

and substituting 1 for the variable z (i.e. applying the augmentation $\mathcal{C}(1) \to \mathcal{C}(0)$), we get the unit, but when starting with element of degree $< k$, the corresponding part of the $\mathbb{Z}$-sorted operad $\mathcal{C}_k$ is 0, so the identity is 0 on those elements, proving that we have a k-conective vertex algebra.

On the other hand, starting with a k-connected vertex algebra, using the $\mathcal{C}$-co-alegbra structure which we already proved, co-associativity and Theorem 10, we see that (32) specializes to a map

$$V_{k_0} \otimes V_{k_1} \otimes \ldots \otimes V_{k_n} \to \mathcal{C}_k(k_\infty, k_0, \ldots, k_n) \otimes V_{k_\infty},$$

as required. □

4. Constructions of vertex algebras from generators and defining relations

By Theorem 13 and the remarks following Lemma 8, the category of k-connective vertex algebras is equivalent to a category of algebras over an operad, and the equivalence commutes with the forgetful map, so we can speak of free k-connective vertex algebras on a graded vector space and by the remarks preceding Lemma 33, also of k-connective vertex algebras defined by means of generators and defining relations. Note that although we haven't identified an exact vector space basis of a free k-connective vertex algebra $C_k X$ on a graded vector space X, by (90), and Theorem (10), we have a reasonably explicit presentation of $C_k X$ in terms of generators and defining relations in the category of graded vector spaces. The main purpose of this section is to give the presentations of certain well known vertex alegbras by generators and defining relations.

Remark 14. Clearly, the operadic notation is awkward for the purposes of practical applications. From Lemma 2, (or alternately simply Definition 11) we see that every element in a (k-connective) vertex algebra generated by certain elements can be written by finite words using the binary operation

$$a(n)b,\ n \in \mathbb{Z} \tag{45}$$

and unary operation

$$L_{-1}a \tag{46}$$

in the indeterminates a, b. It is appealing from this point of view to rewrite (45) as

$$[a, b]_n \tag{47}$$

and (46) as

$$a'. \tag{48}$$

The following Lemma then suggests that vertex algebras can be understood as a sort of "deformation" of the notion of Lie algebras. Indeed, the Lemma for example clearly exhibits the known fact that for a vertex algebra V, $V/L_{-1}V$ is a Lie algebra with respect to the operation $[?, ?]_0$. Despite of the appeal of the notation of Lemma 15, however, the standard notation (45), (46), and especially the field notation, is often preferable in obtaining intuition from mathematical physics.

Lemma 15. *In a vertex algebra, we have*

$$[a', b]_n = n[a, b]_{n-1}, \tag{49}$$

$$[a, b]'_n = [a', b]_n + [a, b']_n, \tag{50}$$

$$[b, a]_m = (-1)^{m+1} \sum_{j \geq 0} \frac{1}{j!}[a, b]^{(j)}_{j+m}, \tag{51}$$

$$[a, [b, c]_n]_m - [b, [a, c]_m]_n = \sum_{i \geq 0} \binom{m}{i} [[a, b]_i, c]_{m+n-i}. \tag{52}$$

Proof: (52) is an immediate rewrite of (28). (49), (50) are usual properties of the shift in vertex algebras, proved for example in [11]. (51) is a consequence of those properties and (52). □

In figuring out the structure of a vertex algebra in terms of generators and defining relations, it is usually key to understand first the relations among the operations (45) for $n \geq 0$. This corresponds, in physical language, to figuring out the operator product expansion (OPE) of the generators. In many cases, the Existence theorem (see [5], Kac [11], Theorem 4.5) then determines explicitly the structure of the vertex algebra in question.

More precisely, we have:

Proposition 16. *Consider an m-connected vertex algebra given by an ordered set of generators B and relations equating every (45) for $a, b \in B$, $n \geq 0$, $a \leq b$ with a linear combination of derivatives of elements of B, then by Proposition 52,*

$$b_1(n_1)....b_k(n_k)(1) \tag{53}$$

with $b_i \in B$, $0 > n_1 \geq ... \geq n_k$, $n_i = n_{i+1} \Rightarrow b_i \geq b_{i+1}$ generates V as a vector space.

□

Example 17. [6] Let L be any Lie algebra with a 2-cocycle represented as an invariant symmetric bilinear form $\langle ?, ? \rangle$. Then the Lie vertex algebra is the 0-connective vertex algebra VL with generators L in dimension 1, and relations

$$[a, b]_0 = [a, b], \tag{54}$$

$$[a, b]_1 = \langle a, b \rangle \cdot 1. \tag{55}$$

Then for dimensional reasons, no further operations (45) with $n \geq 0$ are possible, and hence by (52), if we denote by B an ordered basis of L, then (53) generates VL, and by the existence theorem (Theorem 4.5 of [11],) (53) is in fact a vector space basis of VL.

Example 18. The Virasoro vertex algebra in the category of 0-connective vertex algebras has one generator L of dimension 2 and relations

$$[L, L]_0 = L', \ [L, L]_1 = 2L, \ [L, L]_2 = 0, \ [L, L]_3 = (c/2) \cdot 1 \tag{56}$$

where c is the central charge. Again, this closes the OPE algebra, and therefore again a vector space set of generators of the Virasoro vertex algebra is (53) where $B = \{L\}$. Again, this is a basis (cf. Frenkel [6]) by the Existence theorem.

Sometimes, the OPE is not sufficient for determining the structure of the vertex algebra completely when there are relations between the higher derivatives of the generators.

Example 19. Let L be an even positive-definite lattice with $\mathbb{Z}/2$-valued bilinear form B such that for all $x \in L$,

$$b(x, x) = \langle x, x \rangle / 2. \tag{57}$$

Then the lattice vertex algebra V_L is

$$VL_{\mathbb{C}} \otimes \mathbb{C}[L] \tag{58}$$

where $L_{\mathbb{C}} = L \otimes \mathbb{C}$ is the trivial Lie algebra with symmetric bilinear form coming from L and $\mathbb{C}[L]$ is the group algebra (we denote the canonical basis elements of $\mathbb{C}[L]$ by (λ), $\lambda \in L$). The $VL_{\mathbb{C}}$ factor is a vertex subalgebra whose structure is specified in Example 17. More generally, we interpret a product $x \otimes (\lambda)$ in (58) as $x(-1)\lambda$. Then in this notation, we have

$$a(0)(\lambda) = \langle a, \lambda \rangle (\lambda), \ \ a(n)(\lambda) = 0, \ n > 0, \ a \in L_{\mathbb{C}}, \tag{59}$$

and for $x \in VL_{\mathbb{C}}$,

$$(\lambda)(z)x(-1)(\mu) = (-1)^{b(\lambda,\mu)}(: \exp(\partial_z^{-1}\lambda(z)) : x)(-1)(\lambda + \mu) \tag{60}$$

where the antiderivative is interpreted in the usual way, cf. [7].

We can therefore take the generators of V_L to be

$$\lambda_{\mathbb{C}}, (\lambda), \tag{61}$$

where $\lambda \in L$. The formulas (59), (60) certainly determine the OPE of the generators (61). More concretely, recall the convention that we write $a_+(z)$ resp. $a_-(z)$ for the sum of terms with non-negative resp. negative power of z in $a(z)$, then the corresponding *normal order* expression is

$$: a(z)b(t) := a_+(z)b(t) + b(t)a_-(z).$$

Then we have

$$(\lambda)(z)(\mu)(t) =: (\lambda(z)(\mu)(t) : +(z-t)^{\langle\lambda,\mu\rangle}(\lambda+\mu)(t), \tag{62}$$

$$\mu_{\mathbb{C}}(z)(\lambda)(t) =: \mu_{\mathbb{C}}(z)(\lambda)(t) : +\langle\lambda,\mu\rangle(z-t)^{-1}(\lambda)(t), \tag{63}$$

$$\mu_{\mathbb{C}}(z)\lambda_{\mathbb{C}}(t) =: \mu_{\mathbb{C}}(z)\lambda_{\mathbb{C}}(t) : +\langle\lambda,\mu\rangle(z-t)^{-2}. \tag{64}$$

When considering operations of the form (47) with $n \geq 0$, then the normal order expression vanishes. Therefore, clearly the relations we get from (61) satisfy the assumptions of Propositiion 16 if we choose the order in such a way that

$$\lambda_{\mathbb{C}} < (\mu)$$

for any $\lambda, \mu \in L$.

Even then, however, we see that there are more expressions (53) than a basis of (58), since (58) allows only one label generator, and no derivatives. We can remedy the situation by including the relation

$$(0) = 1$$

and a relation following from (60) which expresses any $\lambda(n)\mu$, $n < 0$ as a product of terms of the form $\mu_{\mathbb{C}}(m)$, $m < 0$ from the left with $(\lambda + \mu)$.

Nevertheless, in this case, there is an extremely clever clever solution given by the following

Proposition 20. [15] *Let $\mathbb{P}^\infty$ be a basis of the lattice L, and assume that $b(x,x) = 1$ for $x \in \mathbb{P}^\infty$. Then the in the category of* 0*-connective vertex algebras, V_L can be presented in generators*

$$(\lambda),\ \lambda \in \pm\mathbb{P}^\infty,$$

and relations

$$(\lambda)(n)(\mu) = 0 \text{ when } n < -\langle\lambda,\mu\rangle,$$
$$(\lambda)(||\lambda||^2)(-\lambda) = 1.$$

Proof: Roitman [15], Section 10 proves that the canonical morphism

$$\psi : V \to V_L \tag{65}$$

from the vertex algebra V given by the given generators and defining relations in his category to V_L is an isomorphism. Therefore, V is in particular 0-connective. However, since the 0-connectivity restriction can be viewed as additional relations in Roitman's category, (65) factors through a morphism

$$\tau : V \to V_0 \tag{66}$$

where V_0 is the 0-connected vertex algebra with the given generators and defining relations. Since, in the present case, ψ is iso, so is τ. □

One canonical way of always selecting relations is as follows: let V be a k-connective vertex algebra, $k \leq 0$. Consider then the relations

$$\begin{array}{l} a = 0 \text{ for } a \in V \text{ whenever } |b_1(n_1)...b_k(n_k)a| \leq 0 \text{ implies} \\ b_1(n_1)...b_k(n_k)a = 0 \text{ for } b_1, ..., b_k \in V,\ n_1, ..., n_k \in \mathbb{Z}. \end{array} \tag{67}$$

Lemma 21. *As a vector space, the quotient of V by the relations (67) is isomorphic to V/I where*

$$\begin{array}{l} I = (L_{-1})^m(b_1(n_1)...b_k(n_k)a) \text{ where } b_i \in V,\ n_i \in \mathbb{Z} \text{ and } a \\ \text{is as in (67), } m \geq 0. \end{array} \tag{68}$$

In particular,

$$V/I \neq 0. \tag{69}$$

Proof: By Lemma 33, the map from V to the quotient by the relations (67) is onto. Further, by its definition, I is clearly contained in the kernel. Thus, by universality it suffices to show that V/I is indeed a vertex algebra, or that I is an ideal in the category of k-connective vertex algebras considered as universal algebras. This means that I must be stable under all operations where precisely one entry is in I and the other entries are in V. Clearly, I is stable under the derivative. Additionally, clearly $a \in V$ and $u \in I$ together imply $a(n)u \in I$, by (49) and (50). It then follows that also $u(n)a \in I$ by (51). The assertion (69) now clearly follows from the condition on a in (67). □

We shall refer to the ideal I from Lemma 21 as the *maximal positive ideal*. The quotients of Lie vertex algebras (resp. Virasoro vertex algebras) by the maximal positive ideal I, when $I \neq 0$, are referred to as WZW models (resp. minimal models). (There are variants of all these notions involving supersymmetry, but we won't discuss them here.) One can also prove that the lattice vertex algebra can be characterize in this way:

Proposition 22. *Let V be the vertex algebra with generators (61) and relations for $a(n)b$ where $n \leq 0$, and a, b are generators, given by the corresponding summands of (60), and let I be the maximal positive ideal in V. Then V/I is the lattice algebra corresponding to L.*

Proof: The key point is to show that the maximal positive ideal of V_L is 0; then there exists a map of 0-connective vertex algebras

$$V/I \to V_L$$

which in turn must be an iso, because its kernel would be contained in the maximal positive ideal of V/I, which must be 0.

To show that the maximal positive ideal of V_L is 0, we invoke the well known fact [7] that V_L is semisimple and its irreducible modules correspond to elements of L'/L (where L' is the dual lattice). For any semisimple vertex algebra, the maximal positive ideal is 0 (for example by Zhu [18]). □

Example 23. The Moonshine module $V^\natural$. Let B be Griess' commutative non-associative algebra [9]. Then B is the weight 2 summand of the Moonshine module [7], and the operation in B is $[?,?]_1$. The operation $[?,?]_2$ is 0 and the operation $[?,?]_3$ is the pairing invariant under the action of the Monster. We can take the free 0-connected vertex algebra on the generator space B with these relations. Let I be the maximal positive ideal in V. Then one has

$$V/I \cong V^\natural. \tag{70}$$

The reason is similar as in the lattice case: $V^\natural$ is semisimple and has only one irreducible module, so its maximal positive ideal is 0. This induces a map from the left hand side to the right hand side of (70). The map is onto because $V^\natural$ is generated by its elements of weight 2 (see [7]). Any kernel would again be contained in the maximal positive ideal of V/I, which is 0.

Determining completely the OPE of the Moonshine module in the sense of Proposition 16 is an interesting problem. This is because we have the operation

$$[?,?]_0. \tag{71}$$

In fact, as we remarked above, for any vertex algebra V, $V/L_{-1}V$ forms a Lie algebra under the operation (71). In the case of $V^\natural$, it is an interesting problem to determine the structure of this Lie algebra L: the weight 3 part of $V^\natural$ decomposes as a representation of the Monster into dimensions

$$21493760 = 21296876 + 196883 + 1.$$

The second two terms are the image of L_{-1}. Thus, we see that L_3 is the irreducible representation of dimension 21493760 and L_2 is the irreducible representation of dimension 196883. In particular, the Lie bracket exhibits the 21493760-dimensional representation as a quotient of the second exterior power of the 196883-dimensional representation.

If we knew the complete structure of L, then we would know the OPE in the sense of Proposition 16; the operations $[?,?]_1$ and (71) satisfy the relations

$$[a,[b,c]_1]_0 - [b,[a,c]_0]_1 = [[a,b]_0,c]_1,$$

$$[a,[b,c]_0]_1 - [b,[a,c]_1]_0 = [[a,b]_0,c]_1 + [[a,b]_1,c]_0,$$

which are special cases of (52).

An easier approach to finding a complete representation of the 0-connective vertex algebra $V^\natural$ by generators and defining relations seems to be from the construction of Frenkel, Lepowsky and Meurman [7]. They constructed $V^\natural$ as the sum of the vertex algebra V_L^θ where L is the Leech lattice and θ is the involution $\lambda \mapsto -\lambda$ and $V_{L,T}^\theta$ where $V_{L,T}$ is the twisted module of V_L with respect to the involution θ. Let us focus on the untwisted sector. We can choose generators

$$\mu_{\mathbb{C}}\nu_{\mathbb{C}}, \tag{72}$$

$$(\lambda) + (-\lambda), \tag{73}$$

$$\mu_{\mathbb{C}}(\lambda) - \mu_{\mathbb{C}}(-\lambda) \tag{74}$$

where $\lambda, \mu, \nu \in L$. These generators do not close under the OPE operations of Proposition 16. However, we can express a general element of V_L^θ as a product

$$b_1(n_1)...b_k(n_k)a \tag{75}$$

where $n_i < 0$, b_i are of the form (72) and a is of the form (73) or (74). Using (64), we can see that the elements (72) close under the OPE, and hence the elements b_i in (75) can be reordered using their OPE relations. Using (63) and (64), one can also write relations equating the appropriate elements

$$b(n)a,\ n<0 \tag{76}$$

where b is of the form (72) and a is of the form (73) (corresponding to reordering the individual $\mu_{\mathbb{C}}$ factors in (64). Finally, using (62) - (64) again, we can describe relations for applying (45) with a of the form (73) or (74) where $||\lambda|| = 2$ to (72)-(74) and yielding results of the form (75). We omit the details. Putting all these relations together, we obtain a presentation of V_L^θ in terms of generators and defining relations. Adding relations describing products (45) where a, b are weight 2 elements of V_T^θ, we could write down an explicit presentation of $V^\natural$ in terms of generators and defining relations.

5. Appendix: Some facts on categories and operads

Universal algebras with underlying structure of a vector space where the additional operations are distributive can be often axiomatized as algebras over an operad. Co-operads become useful when certain finiteness conditions fail. Because of variations in the definitions, we recall here the definition of an operad, co-operad, and universal algebra over them. We will work in the underlying category $Vect$ of $\mathbb{C}$-vector spaces and homomorphisms. We will also consider the category $\mathbb{Z} - Vect$ of $\mathbb{Z}$-graded $\mathbb{C}$-vector spaces and homomorphisms preserving degree.

Definition 24. An *operad* $\mathcal{C}$ in $Vect$ is a sequence of objects $\mathcal{C}(n)$, $n = 0, 1, 2, ...$ with right Σ_n-action, and morphisms

$$\gamma : \mathcal{C}(n) \otimes \mathcal{C}(m_1) \otimes ... \otimes \mathcal{C}(m_n) \to \mathcal{C}(m_1 + ... + m_n) \tag{77}$$

such that we have equivariance, stating that for permutations $\tau \in \Sigma_n$, $\sigma_i \in \Sigma(m_i)$,

$$\begin{array}{ccc}
\mathcal{C}(n) \otimes \mathcal{C}(m_1) \otimes ... \otimes \mathcal{C}(m_n) & & \\
\downarrow{\scriptstyle Id\otimes\sigma_1\otimes...\otimes\sigma_n} & \searrow{\scriptstyle \tau\wr\sigma_1,...,\sigma_n} & \\
\mathcal{C}(n) \otimes \mathcal{C}(m_1) \otimes ... \otimes \mathcal{C}(m_n) & & \\
\downarrow{\scriptstyle \tau\otimes T} & & \\
\mathcal{C}(n) \otimes \mathcal{C}(m_{\tau(1)}) \otimes ... \otimes \mathcal{C}(m_{\tau(n)}) & \xrightarrow{\gamma} & \mathcal{C}(m_1 + ... + m_n).
\end{array} \tag{78}$$

($\wr$ denotes the wreath product of permutations and T is the switch) and associativity, which states that

$$\begin{array}{ccc}
\mathcal{C}(n)\otimes M\otimes L_1\otimes ...\otimes L_n & \xrightarrow{\gamma\otimes Id\otimes ...\otimes Id} & \mathcal{C}(m_1+...+m_n)\otimes L_1\otimes ...\otimes L_n \\
\Big\downarrow {\scriptstyle Id\otimes\gamma\otimes ...\otimes\gamma} & & \Big\downarrow {\scriptstyle \gamma} \\
\mathcal{C}(n)\otimes\mathcal{C}(\ell_1)\otimes ...\otimes\mathcal{C}(\ell_n) & \xrightarrow[\gamma]{} & \mathcal{C}(\ell_1+...+\ell_n)
\end{array} \tag{79}$$

where $\ell_i=\ell_{i1}+...+\ell_{ik_i}$,

$$\mathcal{M}=\mathcal{C}(m_1)\otimes\mathcal{C}(m_2)\otimes\mathcal{C}(m_n),$$

$$L_i=\mathcal{C}(\ell_{i1})...\otimes\mathcal{C}(\ell_{ik_i}).$$

Since (78), (79) are expressed in terms of diagrams, we may therefore define a *co-operad* as a sequence of objects $\mathcal{C}(n)$, $n=0,1,2,...$, a left Σ_n action on $\mathcal{C}(n)$ and morphisms

$$\gamma:\mathcal{C}(m_1+...+m_n)\to\mathcal{C}(n)\otimes\mathcal{C}(m_1)\otimes ...\mathcal{C}(m_n) \tag{80}$$

such that diagrams dual to (78) and (79) commute. Definitions of operads and co-operads over $\mathbb{Z}-Vect$ are the same, except we work in the category $\mathbb{Z}-Vect$. Specifically, a $\mathbb{Z}$-graded co-operad is a collection of spaces $\mathcal{C}(n)_k$, $k\in\mathbb{Z}$ and for $k=k_1+...+k_n$, maps

$$\gamma:\mathcal{C}(m_1+...+m_n)_k\to\mathcal{C}(m_1)_{k_1}\otimes ...\otimes\mathcal{C}(m_n)_{k_n} \tag{81}$$

which satisfy the obvious analogue of the axioms (78), (79).

Remark 25. First note that one could conceivably imagine a stronger axiom than (81) in the case of graded co-operads, as (81) allows an "infinite sum" when one varies the degrees $k_1,...,k_n$ with given sum k on the right hand side.

Next, note that $\mathbb{Z}$-graded operads and co-operads model situations when we have operations which shift total degree by a given number, but the operations which apply to each tuple of degrees are exactly the same. When we want to capture the situation where available operations depend on the input degrees, we must introduce the notion of *$\mathbb{Z}$-sorted operad*. This is a system of vector spaces $V(m_0,m_1,...,m_n)$, $m_i\in\mathbb{Z}$ where the right action of $\sigma\in\Sigma_n$ has

$$\sigma:V(m_0,m_1,...,m_n)\to V(m_0,m_{\sigma(1)},...,m_{\sigma(n)})$$

and composition is defined as

$$\gamma:V(m_0,m_1,...,m_n)\otimes V(m_1,m_{11},...,m_{1,k_1})\otimes ...\otimes V(m_n,m_{n1},...,m_{n,k_1})\to$$
$$V(m_0,m_{11},...,m_{nk_n}).$$

The diagram axioms (78), (79) are the same. Because the axioims are again in shapes of diagrams, we have a dual notion of $\mathbb{Z}$-sorted co-operads. In fact, here, one may replace $\mathbb{Z}$ by any set I, and define I-sorted operads (or co-operads) in the same fashion.

Proposition 26. *The forgetful functor from the category of operads (and the obvious homomorphisms - maps preserving the operations) to vector spaces is a right adjoint (similarly for the forgetful functor from graded and $\mathbb{Z}$-sorted operads to the category of $\mathbb{N}\times\mathbb{Z}$-graded vector spaces and vector spaces graded by $\coprod_{n\geq 0}\mathbb{Z}^n$). The forgetful functor from the category of co-operads (resp. graded co-operads, resp. $\mathbb{Z}$-sorted co-operads) to $\mathbb{N}$-graded vector spaces (resp. the category of $\mathbb{N}\times\mathbb{Z}$-graded vector resp. vector spaces graded by $\coprod_{n\geq 0}\mathbb{Z}^n$) is a left adjoint.*

We shall postpone the proof because it is an example of an even more general principle.

Definition 27. Let $\mathcal{C}$ be an I-sorted operad. A *$\mathcal{C}$-algebra* is a system of vector spaces X_i, $i\in I$, and a system of maps

$$\theta:\mathcal{C}(i_0,i_1,...,i_n)\otimes X_{i_1}\otimes...\otimes X_{i_n}\to X_{i_0} \tag{82}$$

satisfying

$$(\theta\sigma)(?,...?)=\theta(\sigma(?,...,?)) \tag{83}$$

where $\theta\in\Sigma_n$ and σ acts on tuples by permutation, and

$$\theta(\gamma(?,?,...,?),?,...,?)=\theta(?,\theta(?,?,...,?),...,\theta(?,?,...,?)) \tag{84}$$

when applicable.

Remark 28. In the previous definition, one may also define an endomorphism operad and coendomorphism operad of X by

$$End(X)(i_0,i_1,...,i_n)=Hom(X_{i_1}\otimes...\otimes X_{i_n},X_{i_0}), \tag{85}$$

$$Coend(X)(i_0,i_1,...,i_n)=Hom(X_{i_0},X_{i_1}\otimes...\otimes X_{i_n}). \tag{86}$$

Then a $\mathcal{C}$-algebra is the same thing as a homomorphism of I-sorted operads

$$\mathcal{C}\to End(X). \tag{87}$$

It is therefore natural to define a coalgebra X over an I-sorted operad $\mathcal{C}$ to be a morphism of operads

$$\mathcal{C}\to Coend(X). \tag{88}$$

This, of course can also be written in terms of maps

$$\theta:\mathcal{C}(i_0,i_1,...,i_n)\otimes X_{i_0}\to X_{i_1}\otimes...\otimes X_{i_n} \tag{89}$$

and conditions analogous to (83), (84).

We can also define algebras resp. coalgebras over a co-operad by dualizing (82), (83), (84) resp. (89) and the corresponding conditions. However, in for these dual notions,we do not know of an analogue of the descriptions (87), (88).

Proposition 29. *For every set I, there exists an $\coprod_{n\geq 0}I^{n+1}$-sorted operad Q_I such that the category of I-sorted operads (resp. co-operads) is canonically equivalent to the category of Q_I-algebras (resp. Q_I-co-algebras).*

□

Proposition 26 now follows from the following result.

Proposition 30. *Let $\mathcal{C}$ be an I-sorted operad. Then the forgetful functor from the category of $\mathcal{C}$-algebras (resp. $\mathcal{C}$-coalgebras) to I-graded vector spaces is a right (resp. left) adjoint.*

Proof: The statement about $\mathcal{C}$-algebras is classical. In effect, the left adjoint is

$$CX_i = \bigoplus_{n\geq 0} \Big(\bigoplus_{(i,i_1,...,i_n)\in I^{n+1}} \mathcal{C}(i,i_1,...,i_n)\otimes X_{i_1}\otimes ...\otimes X_{i_n}\Big)/\Sigma_n, \tag{90}$$

and in fact the category of $\mathcal{C}$-algebras is equivalent to the category of algebras over the monad C. (For the definition of monads and their algebras, see [12].)

The statement about $\mathcal{C}$-coalgebras follows from the fact that the forgetful functor from $\mathcal{C}$-coalgebras to I-graded vector spaces obviously preserves coproducts and coequalizers, and one can use the adjoint functor theorem to show it must have a right adjoint. □

Remark 31. It is not true that the right adjoint of the forgetful functor from $\mathcal{C}$-coalgebras to I-graded vector spaces would be given by a dual of (91). The dual is

$$PX = \prod_{n\geq 0} \Big(\prod_{(i,i_1,...,i_n)\in I^{n}} Hom(\mathcal{C}(i,i_1,...,i_n), X_{i_1}\otimes ...\otimes X_{i_n})^{\Sigma_n}, \tag{91}$$

but (91) fails to be a comonad. The problem is that tensor product does not distribute under the product, so when one writes the formula for a map

$$PX \to PPX$$

dual to the classical monad structure on C, the image actually will be a vector space containing but not equal to PPX. The actual comonad is the intersection $P'X$ of the inverse images of all such maps $X \to P^n X$.

Remark 32. Another example of failure of dualization is the fact that the forgetful functor from the category of algebras over an I-sorted cooperad $\mathcal{C}$ to the category of I-graded vector spaces does *not* in general create products. Note that the structure map is

$$X_{i_1}\otimes ...X_{i_n} \to \mathcal{C}(i,i_1,...,i_n)\otimes X_i. \tag{92}$$

When we take a product of $\mathcal{C}$-coalgebras $X(j)$, then the image of the structure map (92) for $\prod X(j)$ ends up in

$$\prod \mathcal{C}(i,i_1,...,i_n)\otimes X(j)_i,$$

while $\mathcal{C}$-coalgebra structure requires it to be in the subspace

$$\mathcal{C}(i,i_1,...,i_n)\otimes \prod X(j)_i.$$

However, note that when all of the spaces $\mathcal{C}(i,i_1,...,i_n)$ are finite-dimensional, then the collection of the dual spaces $\mathcal{C}(i,i_1,...,i_n)^\vee$ forms an I-sorted operad $\mathcal{C}^\vee$, and the category of $\mathcal{C}$-algebras is equivalent to the categort of $\mathcal{C}^\vee$-algebras, which, as remarked above, is canonically equivalent to the category of monads over the category of I-graded vector spaces. In particular, in that case, the forgetful functor does create products.

Let $\mathcal{C}$ be an I-sorted operad. There is one more construction we need to cover, namely quotient $\mathcal{C}$-algebras, and the dual construction for $\mathcal{C}$-coalgebras. The most general version of the story is this: Let

$$U : C \to D \tag{93}$$

be a functor with left adjoint L, and let X be an object of C. Then we can consider the category X/C of arrows $X \to ?$, where ? is an object or morphism of C. The forgetful functor

$$X/C \to C$$

creates limits. Then U induces a functor

$$X/U : X/C \to UX/D, \tag{94}$$

which therefore preserves limits, and by the adjunct functor theorem has a left adjoint X/L. Similarly, when (93) has a right adjoint R, and X is an object of C, we may consider the category C/X of arrows $? \to X$, and the forgetful functor

$$U/X : C/X \to D/UX \tag{95}$$

preserves colimits, and hence has a right adjoint R/X by the adjoint functor theorem. Note that this construction applies to the cases of $\mathcal{C}$-algebras and $\mathcal{C}$-coalgebras by Proposition 30.

Lemma 33. *When D is the category of I-graded vector spaces and C is the category of algebras (resp. coalgebras) over an I-sorted operad, U is the forgetful functor (see (93)) and L (resp. R) is the left (resp. right) adjoint, then the functor X/L (resp. R/X) preserves epimorphisms (resp. monomorphisms).*

Proof: The key point is that for a morphism

$$f : A \to B \tag{96}$$

of $\mathcal{C}$-algebras (resp. $\mathcal{C}$-coalgebras), the image of f is a $\mathcal{C}$-algebra (resp. $\mathcal{C}$-coalgebra). This factorizes any morphism (96) as

$$f = gh \text{ where } g \text{ is mono and } h \text{ is epi.} \tag{97}$$

But if we take an epimorhpism (resp. monomorphism) ϕ in UX/D (resp. D/UX), and consider $f = X/L(\phi)$, then factoring f as in (97), h (resp. g) enjoys the same universal property as f, and hence must be isomorphic to f. $\square$

Remark 34. In the case of $\mathcal{C}$-algebras, we can interpret $X/L(\phi)$ where ϕ is an epimorphism of I-graded vector spaces as the quotient of X by the ideal generated by $Ker(\phi)$. In the case of $\mathcal{C}$-coalgebras, a the interpretation is dual.

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