

# A UNIVERSAL APPROACH TO VERTEX ALGEBRAS

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## 1. INTRODUCTION

The notion of vertex algebra due to Borcherds [3] and Frenkel-Lepowsky-Meurman [8] is one of the fundamental concepts of modern mathematics. The definition is purely algebraic, but its central point is an intricate Jacobi identity ([8]), which makes the definition hard to motivate from first principles. The identity can be replaced by a locality axiom [5], cf. [7, 13], see also (24) below, which however feels more like a part of physics than algebra. Of course, vertex algebras are an algebraic model a part of conformal field theory, which enters into the motivation, and also suggests generalizations (cf. Borcherds [4], Soibelman [19] and Beilinson-Drinfeld [1]).

In this paper, we characterize vertex algebras by exploring the algebraic structure present on the *correlation functions* of a vertex algebra. This turns out to be a structure of a co-operad, the dual of the notion of operad. Operads are known to play an important role in algebra (cf. Ginzburg-Kapranov [9]) and topology (cf. May [15]). In algebra, operads describe vector spaces with additional multilinear operations, which are subject to universal multilinear identities. Our axiomatization of vertex algebra via the co-operad structure on correlation functions does, in a way, explain the Jacobi identity within the confines of algebra. We stress that our use of operads is different from the paper of Huang and Lepowsky [12], whose operads (or partial operads) consist of points, while our co-operads consist of functions (in a vague sense, the approaches could be called “Koszul-dual”).

When looking at the picture in more detail, subtleties emerge, dealing with which comprises a large part of this paper. The extant definitions of vertex algebra ([3, 8, 7, 13]) vary slightly; to get a universal algebra treatment, we need to fix notions. We end up focusing on two notions which differ by a degree of *uniformity* of the locality required. In any case, however, if we deal with general (non-connective) vertex algebras, the category lacks the properties of a category of universal algebras, which forces us to use co-operads instead of operads. The algebraical and categorical discussions resulting are in a way standard, but as far as we know, are not in the literature, which is why we include an appendix (Section 5) dealing with the prerequisites.

When we assume connectivity, (a vertex algebra is  $k$ -connective when  $V_n = 0$  for  $n < k$ ), we pass to a sub-co-operad which is finite-dimensional, and thus can be dualized to an actual operad, which we can, at least in a certain sense, describe

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explicitly (Theorem 10 below). Thus,  $k$ -connective vertex algebras for a fixed  $k$  are actual universal algebras, and can be defined in terms of generators and defining relations. In Section 4, we give some examples of using this method, although this is just a beginning, and more work needs to be done. We should also mention the very important work of Roitman who in a series of papers [16, 17, 18] explored the category of vertex algebra restricted by specific bounds of locality between given generators. In his category, Roitman also has a notion of construction from generators and defining relations (in particular, he studies free objects), but his category isn't an actual category of universal algebras, and his results are different from ours. For examples, Roitman's free objects typically have no connectivity, which has a physical explanation in the Coulomb gas realization of conformal field theories (cf. [11]). Still, in certain cases, (when a Roitman presentation enjoys connectivity), presentations in our sense can be deduced from Roitman's (e.g. the case of positive-definite lattices, see Section 4 below). Finally, we should also mention the related work of Bokut, Fong and Ke [2], who studied presentations of conformal algebras, and even developed a Gröbner basis style algorithm in that case.

The present paper is organized as follows: In Section 2, we describe the co-operad of correlation functions and its  $k$ -connective version. In Section 3, we prove our main results characterizing vertex algebras and  $k$ -connective vertex algebras in this setting. Section 4 contains examples of presentations of vertex algebras in terms of generators and defining relations. Section 5 is the Appendix as mentioned above.

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## 2. VA-ALGEBRAS AND THE CORRELATION FUNCTION CO-OPERAD

In this paper, all vector spaces we consider will be over the field of complex numbers  $\mathbb{C}$ . We shall say that a meromorphic function  $f(z)$  on  $\mathbb{C}P^1$  (the one-dimensional complex projective space) is *non-singular at  $\infty$*  when the limit of  $f(z)$  at  $\infty$  is 0.

**Definition 1.** A *local function* is a meromorphic  $n$ -variable function on  $\mathbb{C}P^1$  which is non-singular when all the variables are different and different from  $\infty$ .

The space of all local functions in variables  $z_1, \dots, z_n$  will be denoted by  $\mathcal{C}(z_1, \dots, z_n)$ .

**Comment:** Local functions can also be characterized as regular functions (in the algebraic sense) on the affine variety consisting of ordered  $n$ -tuples of distinct points of the affine space  $\mathbb{A}^1$  (the "ordered configuration space"). Most of our arguments have a parallel in the category of algebraic varieties over any field of characteristic 0 (we will point out where changes are needed).

**Lemma 2.** *There exists a vector space basis  $B_n$  of  $\mathcal{C}(z_1, \dots, z_n)$  such that*

$$(1) \quad B_0 = \{1\},$$

$$(2) \quad B_{n+1} = \{(z_{n+1} - z_i)^k, z_{n+1}^\ell \mid i = 1, \dots, n, 0 > k \in \mathbb{Z} \ 0 \leq \ell \in \mathbb{Z}\} \cdot B_n.$$

**Proof:** Let  $f = f(z_1, \dots, z_{n+1}) \in \mathcal{C}(z_1, \dots, z_{n+1})$ . Then consider  $f$  as a function of  $z_{n+1}$  with  $z_1, \dots, z_n$  constant. Since every holomorphic function on  $\mathbb{C}P^1$  is constant,

there exist unique functions  $g_k^\ell, g_{1k}, \dots, g_{nk}$ ,  $0 > k \in \mathbb{Z}$ ,  $0 \leq \ell \in \mathbb{Z}$ , all but finitely many of which are 0, such that

$$(3) \quad f = \sum_k g_k z_{n+1}^k + \sum_{i,k} g_{ik} (z_{n+1} - z_i)^k.$$

Clearly,  $g_\ell, g_{ik} \in \mathbb{C}(z_1, \dots, z_n)$ , so an induction completes the proof.  $\square$

Clearly, there is a right action of  $\Sigma_n$  on  $\mathcal{C}(z_1, \dots, z_n)$  given, for a permutation  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ , by

$$(4) \quad f\sigma(z_1, \dots, z_n) = f(z_{\sigma(1)}, \dots, z_{\sigma(n)}).$$

We shall also denote  $\mathcal{C}(z_1, \dots, z_n)$  by  $\mathcal{C}(n)$ ,  $n \geq 0$ .

Now note that it follows from Lemma 2 that each of the vector spaces  $\mathcal{C}(n)$  is graded by *the negative of the degree* (a function  $f(z_1, \dots, z_n)$  is homogeneous of degree  $k \in \mathbb{Z}$  if  $f(az_1, \dots, az_n) = a^k f(z_1, \dots, z_n)$ ). Denote by  $\mathcal{C}(n)_k$  the subspace of elements of degree  $k$ . (The sign reversal will be needed when relating our concept to the concept of vertex algebras; in writing fields of vertex algebras, the degree of a map attached as a coefficient at  $z^n$  minus  $n$  is a constant.)

For  $k = p + q$ , we shall now describe a map

$$(5) \quad \Phi_{n,m} : \mathcal{C}(n)_k \rightarrow \mathcal{C}(m+1)_p \otimes \mathcal{C}(n-m)_q.$$

This is done as follows: consider a function  $f \in \mathcal{C}(z_1, \dots, z_n)_k$ . By Lemma 2,  $f$  can be expressed (non-uniquely) as a polynomial in

$$(6) \quad (z_i - z_j)^{-1},$$

$$(7) \quad z_i.$$

Now modify this polynomial as follows: Replace every monomial (6) with  $i > m$ ,  $j \leq m$  by

$$(8) \quad ((t - z_j) + (z_i - t))^{-1},$$

and expand in increasing powers of  $z_i - t$ . Also replace every monomial (7) with  $i > m$  by

$$(9) \quad (t + (z_i - t)).$$

Then substitute  $t_s$  for  $z_{s+m} - t$  with  $s = 1, \dots, n - m$  and  $z_{m+1}$  for  $t$ . In particular, a term (6) will remain unchanged for  $i, j \leq m$ , while for  $i, j > m$ , the term (6) will turn into

$$(t_{i-m} - t_{j-m})^{-1}.$$

Picking out terms of degree  $p$  in the variables  $z_1, \dots, z_{m+1}$ , and degree  $q$  in the variables  $t_1, \dots, t_{n-m}$ , we obtain an element

$$(10) \quad g \in \mathcal{C}(z_1, \dots, z_{m+1})_p \otimes \mathcal{C}(t_1, \dots, t_{n-m})_q$$

(one easily sees that the number of summands of the given fixed degree is finite). Put  $\Phi_{m,n}(f) := g$ .

**Lemma 3.** *The element  $g$  of (10) does not depend on the representation of  $f$  as a polynomial in (6), (7).*

**Proof:** Interpret  $z_1, \dots, z_m$  as constants different from each other and different from  $\infty$ . Pick  $t \in \mathbb{C}P^1$  different from  $z_1, \dots, z_m$  and from  $\infty$ . Then, putting

$$P = \prod_{i \neq j} (z_i - z_j)^N$$

for a sufficiently large integer  $N$ ,  $gP$  can be characterized as the appropriate homogeneous summand of the Taylor expansion of  $fP$  as a function in  $z_{m+1}, \dots, z_n$  in the neighborhood of

$$(z_{m+1}, \dots, z_n) = (t, \dots, t).$$

□

From now on, we will be using notation and certain facts from category theory and the theory of operads. These facts are somewhat technical and independent of the rest of the material, and are treated in the Appendix.

**Theorem 4.** *There exists a unique graded co-operad structure on the sequence  $(\mathcal{C}(n))$  (we call this the correlation function co-operad) such that  $\Sigma_n$ -equivariance is given by (4) and the co-composition operation corresponding to inserting  $n - m$  variables to the  $m + 1$ 'st variable among variables indexed  $1, \dots, m, m + 1$  is given by  $\Phi_{n,m}$ .*

**Proof:** The axioms we then need to verify are the version of equivariance for insertions, commutativity of insertions into two different original variables, and associativity of insertions (arising when an insertion is followed by another insertion into the new variables).

Regarding proving these properties, first note that equivariance is obvious, expressing simply symmetry of the construction in the variables involved in equal capacity. Regarding commutativity of insertions, using (8), it corresponds to the fact that expanding

$$((s - t) + (z_i - s) - (z_j - t))^{-1}$$

in increasing powers of  $(z_i - s)$  and then increasing powers of  $(z_j - t)$  gives the same series as expanding in  $(z_j - t)$  first and in  $(z_i - s)$  second. One readily verifies that the result in both cases is

$$(11) \quad \sum_{m,n \geq 0} \binom{m+n}{n} (-1)^n (z_j - s)^n (z_i - t)^m (s - t)^{-m-n-1}.$$

Coassociativity is equivalent to saying that expanding

$$(-(z_i - s) + (z_j - s))^{-1}$$

in increasing powers of the second summand, and then write  $(z_j - s)$  as

$$((t - s) + (z_j - t))$$

and expand in increasing powers of the second summand gives the same result as expanding

$$(-(z_i - t) + (z_j - t))^{-1}$$

in the second summand and then writing  $-(z_i - t)$  as

$$(-(z_i - s) + (t - s))$$

and expanding in increasing powers of the second summand. Computation shows that both computation yield

$$(12) \quad - \sum_{m,n \geq 0} \binom{m+n}{m} (z_j - t)^n (t - s)^m (z_i - s)^{-n-m-1}.$$

□

**Comment:** In our treatment of the co-operad  $(\mathcal{C}(n))$ , we used the version of the definition mentioned in Remark 25, which simplifies notation considerably. However, it is not difficult to describe the general co-operation from Definition 24: This is a map of the form

$$(13) \quad \begin{aligned} & \mathcal{C}(z_{11}, \dots, z_{1n_1}, \dots, z_{k1}, \dots, z_{kn_k})_{\ell} \\ & \rightarrow \mathcal{C}(z_1, \dots, z_k)_{\ell_0} \otimes \mathcal{C}(t_{11}, \dots, t_{1n_1})_{\ell_1} \otimes \dots \otimes \mathcal{C}(t_{k1}, \dots, t_{kn_k})_{\ell_k} \end{aligned}$$

with  $\ell_0 + \dots + \ell_k = \ell$ . To get this map, start with the variables  $z_{ij}$ , then introduce new variables  $z_i, t_{ij}$ , and replace  $z_{ij}$  with

$$t_{ij} + z_i.$$

The terms which require expansion are

$$(14) \quad (t_{ij} + z_i - t_{i'j'} - z_{i'})^{-1}, \quad i \neq i'.$$

We rewrite (14) as

$$(t_{ij} + (z_i - z_{i'}) - t_{i'j'})^{-1},$$

expanding in increasing powers of  $t_{ij}$  and  $t_{i'j'}$ . The same formulas (11), (12) are involved in showing that this is well defined and associative.

**Definition 5.** A VA-algebra is a graded algebra over the graded co-operad  $\mathcal{C}$ .

We shall also be interested in cutting off VA-algebras by connectivity.

**Definition 6.** A *bounded VA-algebra* is a VA-algebra  $X$  such that for every  $x_1, \dots, x_n \in X$ , the image of

$$x_1 \otimes \dots \otimes x_n$$

under the coalgebra structure map in

$$X \otimes \mathcal{C}(n)_k$$

is 0 for  $k < -N$  where  $N$  depends only on  $x_1, \dots, x_n$ .

Now let  $k \in \mathbb{Z}$ . Consider  $\mathcal{C}$  as a  $\mathbb{Z}$ -sorted co-operad in the obvious way (see Remark 26). Then in particular,

$$(15) \quad \begin{aligned} \mathcal{C}(n, n_1, \dots, n_m) = \mathcal{C}(m)_{n_1 + \dots + n_m - n} \text{ only depends on } n_1 + \\ \dots + n_m - n \end{aligned}$$

(the difference  $n_1 + \dots + n_m - n$  being the degree).

Now consider the  $\coprod_{n \geq 0} \mathbb{Z}^{n+1}$ -sorted vector subspace  $\mathcal{C}'_k$  of  $\mathcal{C}$  which in degree  $(n, n_1, \dots, n_m)$  is equal to  $\mathcal{C}(n, n_1, \dots, n_m)$  when  $n \geq k$ , and 0 when  $n < k$ . By Proposition 30, Lemma 34 and Remark 35, there is a universal largest  $\mathbb{Z}$ -sorted sub-co-operad  $\mathcal{C}_k \subset \mathcal{C}$  contained in  $\mathcal{C}'_k$ .

**Definition 7.** A  $k$ -connective VA-algebra is an algebra over the co-operad  $\mathcal{C}_k$ .

**Lemma 8.** For every  $k \in \mathbb{Z}$ , and every  $n, n_1, \dots, n_m$ , the vector space  $\mathcal{C}_k(n, n_1, \dots, n_m)$  is finite-dimensional.

**Proof:** Suppose the dimension of  $\mathcal{C}_k(n, n_1, \dots, n_m)$  is infinite. Then the degree of singularity of elements at  $z_i \rightarrow z_j$  is unbounded for some  $i \neq j$  (in fact, we may choose  $i = 1, j = 2$  by symmetry). Therefore, by Lemma 2, we must have elements which map by the comultiplication into non-zero elements of

$$(16) \quad \mathcal{C}(n, i, n_3, \dots, n_m) \otimes \mathcal{C}(i, n_1, n_2)$$

with  $i$  arbitrarily low. But for  $i < k$ ,

$$(17) \quad \mathcal{C}_k(n, i, n_3, \dots, n_m) \otimes \mathcal{C}_k(i, n_1, n_2) = 0$$

(since the second factor is 0), which is a contradiction.  $\square$

By Lemma 8, and Remark 33, the category of  $k$ -connective VA-algebras is the category of algebras over an operad, and in particular has free objects and factorization (see Remark 35).

### 3. THE MAIN RESULTS

We begin with a more precise version of Lemma 8, which is more technical, but is proved by exactly the same method. Let  $1 \leq i_1 < \dots < i_p \leq m$ . Let us introduce an increasing filtration on  $\mathcal{C}(z_1, \dots, z_m)$  as follows: For  $N < 0$ , let  $F(i_1, \dots, i_p)_N \mathcal{C}(z_1, \dots, z_m) = 0$ . For  $N \geq 0$ , call  $f \in \mathcal{C}(z_1, \dots, z_n)$   $N$ -regular if the following condition holds: Fix a set  $S$  of points  $z_j \in \mathbb{C}$ ,  $j \neq i_1, \dots, i_p$ . Assume the  $z_j$ 's are all different. Then  $\prod (z_{i_q} - z_{i_s})^{k_{qs}} f$  is holomorphic on  $(\mathbb{C} - S)^p$  for some integers  $k_{qs} \geq 0$ ,  $\sum k_{ij} \leq N$ . , let

$$(18) \quad F(i_1, \dots, i_p)_N \mathcal{C}(z_1, \dots, z_n) := \langle f \in \mathcal{C}(z_1, \dots, z_n) \mid f \text{ is } N\text{-regular} \rangle.$$

**Lemma 9.** For  $N \geq 0$ , there exists a vector space basis  $B(i_1, \dots, i_p)_n^N$  of

$$F(i_1, \dots, i_p)_N \mathcal{C}(z_1, \dots, z_n)$$

such that

$$(19) \quad B(i_1, \dots, i_p)_0^N = \{1\},$$

$$(20) \quad B(i_1, \dots, i_p)_{n+1}^N = \{(z_{n+1} - z_i)^k, z_{n+1}^\ell \mid i = 1, \dots, n, 0 > k \in \mathbb{Z} \ 0 \leq \ell \in \mathbb{Z}\} \cdot B(i_1, \dots, i_p)_n^N.$$

when  $n + 1 \neq i_p$ , and

$$(21) \quad B(i_1, \dots, i_p)_{n+1}^N = \bigcup_{k+M \leq N} \{(z_{n+1} - z_{i_s})^k \mid s = 1, \dots, p-1, 0 > k \in \mathbb{Z}\} \cdot B(i_1, \dots, i_{p-1})_n^M \cup \{(z_{n+1} - z_i)^k, z_{n+1}^\ell, 0 > k, i \neq i_1, \dots, i_{p-1}, 0 \leq \ell \in \mathbb{Z}\} \cdot B(i_1, \dots, i_{p-1})_n^N$$

when  $i_p = n + 1$ .

**Theorem 10.** *If  $k < 0$ , then  $\mathcal{C}_k(n, n_1, \dots, n_m) = 0$  for all  $n, n_1, \dots, n_m$ . In general,  $\mathcal{C}_k(n, n_1, \dots, n_m) \subseteq \mathcal{C}(z_1, \dots, z_m)$  is the subspace  $W_k$  of all  $f$  such that for any  $1 \leq i_1 < \dots < i_p \leq m$ ,*

$$(22) \quad f \in F(i_1, \dots, i_p)_N \mathcal{C}(z_1, \dots, z_m)$$

where

$$(23) \quad N = -k + n_{i_1} + \dots + n_{i_p}.$$

**Proof:** The first statement immediately follows from the second. To prove the second statement, our definition of the insertion operation clearly implies that  $\mathcal{C}_k(n, n_1, \dots, n_m)$  is contained in  $W_k$ . On the other hand, since the insertion operation does not decrease the powers of the differences of the inserted variables involved (i.e. “does not increase singularity in the inserted variables”), the  $W_k$ ’s form a  $\mathbb{Z}$ -sorted sub-co-operad of  $\mathcal{C}$ .  $\square$

Because of variations in the definition, let us define our notion of vertex algebra.

**Definition 11.** A *vertex algebra* is a  $\mathbb{Z}$ -graded vector space  $V$  together with, for each  $a \in V_k$ , a series

$$(24) \quad a(z) = \sum_n a(n)z^{-n-1}$$

where

$$a(n) : V_m \rightarrow V_{m-n+k-1},$$

an operator  $L_{-1} : V_n \rightarrow V_{n+1}$ , and an element  $1 \in V_0$  such that the following properties hold:

$$(25) \quad \text{For every } a, b, c \in V \text{ and every } m \in \mathbb{Z}, \text{ there exists an } N \in \mathbb{N} \text{ such that the } V_m[[z, t]]\text{-summand } s_m(z, t) \text{ of } (a(z)b(t) - b(t)a(z))c \text{ satisfies } (z - t)^N s_m(z, t) = 0.$$

$$(26) \quad [L_{-1}, a(z)] = \partial_z a(z)$$

(the symbol  $\partial_z$  denotes (partial) derivative by  $z$ ),

$$(27) \quad L_{-1}1 = 0,$$

$$(28) \quad 1(z) = 1,$$

$$(29) \quad a(n)1 = 0 \text{ for } n \geq 0.$$

We will call a vertex algebra *uniform* when the axiom (25) is replaced by the following obviously stronger statement:

$$(30) \quad \text{For } a, b \in V, a(z) \text{ and } b(t) \text{ are local in the sense that } (z - t)^N (a(z)b(t) - b(t)a(z)) = 0 \text{ for some integer } N \geq 0 \text{ (possibly dependent on } a, b).$$

**Comment:** The axioms (25), (30) are known as locality axioms. Traditionally (cf. [13]), the stronger axiom (30) are used. In this paper, we consider  $\mathbb{Z}$ -graded vertex algebras. In this context, the axiom (30) has the non-uniform version (25). On the other hand, in [13], one considers vertex algebras without a  $\mathbb{Z}$ -grading, in which case the non-uniform axiom does not appear to make sense. One has the following result:

**Lemma 12.** [16] *A uniform vertex algebra satisfies the following property:*

$$(31) \quad \text{For any } a, b \in V \text{ there exists } K \in \mathbb{Z} \text{ such that } a(i)b = 0 \text{ for } i > K.$$

**Proof:** By Kac [13], a uniform vertex algebra satisfies the identity

$$(32) \quad a(m)b(k) - b(k)a(m) = \sum_{j \geq 0} \binom{m}{j} (a(j)b)(m+k-j).$$

When (30) holds with a uniform  $N$ , then we claim that (31) must hold with  $K = N$ . Indeed, if  $a(i)b \neq 0$  for  $i > N$ , then for  $k+m-i = -1$ ,

$$a(z)b(t) - b(t)a(z)$$

applied to 1 has the summand

$$\sum_{m \in \mathbb{Z}} \binom{m}{i} a(i)b z^m t^{-1-m+i},$$

which is not annihilated by  $(z-t)^N$ .  $\square$

**Theorem 13.** *There is a canonical equivalence between the category of vertex algebras (resp. uniform vertex algebras, resp. vertex algebras  $V$  with  $V_n = 0$  for  $n < k$ ) and VA-algebras (resp. bounded VA-algebras resp.  $k$ -connective VA-algebras). Moreover, the equivalence commutes with the forgetful functors to  $\mathbb{Z}$ -graded vector spaces.*

**Comment:** By “canonical” we mean that there is an obvious preferred choice for the equivalence (rather than just that an equivalence exists).

**Proof:** First, assume  $V$  is a vertex algebra according to the Definition 11. Then generalizing the usual arguments (cf. [13, 7]), one can show that  $V$  possesses correlation functions. In one way, an  $n+2$ -point correlation function can be described, for any integers  $k_0, k_1, \dots, k_n, k_\infty$  as a linear combination

$$(33) \quad \phi(a_0, a_1, \dots, a_n)(0, z_1, \dots, z_n) = \sum b_j f_j(0, z_1, \dots, z_n)$$

with coefficients  $b_j \in V_{k_\infty}$  of local functions (in the sense of definition 1) in variables  $z_0, z_1, \dots, z_n$ , dependent  $n$ -linearly on homogeneous elements  $a_i \in V_{k_i}$ ,  $i = 0, \dots, n$ , such that  $\phi$  has expansion

$$(34) \quad a_n(z_n) \dots a_1(z_1)(a_0)$$

convergent in the range

$$|z_n| > \dots > |z_1| > 0.$$

Additionally, one can choose

$$(35) \quad f(z_0, z_1, \dots, z_n) = \exp(-z_0 L_{-1}) f(0, z_1 - z_0, \dots, z_n - z_0).$$

The structure map

$$(36) \quad V_{k_0} \otimes V_{k_1} \otimes \dots \otimes V_{k_n} \rightarrow \mathcal{C}(z_0, z_1, \dots, z_n)_{k_\infty - k_0 - \dots - k_n} \otimes V_{k_\infty}$$

is then defined by

$$(37) \quad a_0 \otimes a_1 \otimes \dots \otimes a_n \mapsto \phi(a_0, a_1, \dots, a_n)(z_0, z_1, \dots, z_n) := \sum b_j f_j(z_0, \dots, z_n).$$

Next, we show that this defines a co-operad algebra structure. First, equivariance is a well known property (see e.g. Kac [13]). To prove compatibility with the insertion operator, one can use the well known fact that

$$\phi(a_0, a_1, \dots, a_n)(z_0, z_1, \dots, z_n)$$

also has an expansion

$$(a_n(z_n - t) \dots a_{m+1}(z_{m+1} - t) a_m)(t) a_{m-1}(z_{m-1}) \dots a_1(z_1) a_0(z_0) 1$$

convergent for

$$\begin{aligned} 1 &> |t| > |z_{m-1}| > \dots > |z_1| > |z_0|, \\ |t| - |z_{m-1}| &> |z_n - t| > \dots > |z_{m+1} - t| > 0. \end{aligned}$$

In the uniform case, to prove boundedness, use Lemma 2, Lemma 12 and axiom (31) repeatedly.

On the other hand, assuming  $V$  has the structure of a graded  $\mathcal{C}$ -algebra, and letting, under (36), for  $n = 1$ , as above,

$$(38) \quad a_0 \otimes a_1 \mapsto \phi(a_0, a_1)(z_0, z_1),$$

we let the degree  $k_\infty$  summand of

$$a_1(z)(a_0)$$

be the Taylor expansion at 0 of

$$\phi(a_0, a_1)(0, z).$$

Let also, as usual,  $L_{-1}a$  be defined as the constant term of

$$(a(z)1)'$$

To prove locality, first note that by the fact that a meromorphic function on  $\mathbb{C}P^1$  is the sum of the singular parts of its expansions at all points (this, in turn, is due to the fact that a non-singular function on all of  $\mathbb{C}P^1$  is 0). Now the co-composition axiom and the identity

$$(39) \quad z^m = \sum_{i \geq 0} \binom{m}{i} (z-t)^i t^{m-i} \text{ for } |z-t| < |t|$$

imply that

$$(40) \quad z^m \phi(?, a, b)(0, z, t)$$

is anywhere on  $\mathbb{C}P^1$  equal to

$$(41) \quad \begin{aligned} &\sum_{p \geq m} b(t) a(p) z^{m-p-1} + \sum_{p < m} a(p) b(t) z^{m-p-1} \\ &+ \sum_{0 \leq i \leq j} \binom{m}{i} (z-t)^{i-j-1} (a(j)b)(t) t^{m-i}. \end{aligned}$$

Subtracting  $z$  times (40) from the analogous expression obtained by replacing  $m$  by  $m+1$ , we obtain

$$(42) \quad a(m)b(t) - b(t)a(m)$$

on the left hand side and

$$(43) \quad \sum_{0 \leq i \leq j} \binom{m+1}{i} (z-t)^{i-j-1} (a(j)b)(t) t^{m+1-i} \\ - ((z-t) + t) \sum_{0 \leq i \leq j} (z-t)^{i-j-1} (a(j)b)(t) t^{m-i}.$$

Using

$$\binom{m+1}{i} = \binom{m}{i} + \binom{m}{i-1} \text{ for } i > 0,$$

after chain cancellation, the only surviving term of (43) is the term corresponding to  $\binom{m}{i-1}$  and  $i = j + 1$  in the first summand, i.e.

$$(44) \quad \sum_{0 \leq i} \binom{m}{j} (a(j)b)(t) t^{m-j}$$

Equating (42) and (44) and taking the coefficient at  $t^{-k-1}$  gives (32), which is a part of the Jacobi identity. Even more importantly for us, however, the graded co-operad algebra structure implies that when applied to an element  $c$ , the summand of given degree  $m$  on the right hand side of (32) has  $j$  bounded above, say, by  $N$ . Grouping terms of (32) with  $m + k$  constant, we then see that the  $V$ -degree  $m$  summand of

$$(45) \quad [a(z), b(t)]c$$

is equal to a sum of terms which are  $V$ -degree  $m$  summands of some constant times a power of  $z$  times  $(a(i)b)(t)c$  times the  $i$ 'th derivative of the delta function  $\delta(z/t)$  (say, by  $z$ ), for  $0 \leq i \leq N$ . Thus, (45) is annihilated when multiplied by  $(z-t)^N$ , as required.

One also sees clearly that in the bounded case, the terms of (32) with  $j > N$  for a constant  $N$  dependent on  $a, b$  vanish outright, which implies uniformity.

To prove the axiom (26), consider the following diagram from the definition of algebra over a co-operad:

$$(46) \quad \begin{array}{ccc} X \otimes X & \xrightarrow{\theta} & X \otimes \mathcal{C}(2) \\ \downarrow \theta & & \downarrow \theta \otimes 1 \\ X \otimes \mathcal{C}(2) & \xrightarrow[1 \otimes \gamma]{} & X \otimes \mathcal{C}(1) \otimes \mathcal{C}(2). \end{array}$$

Our plan is to consider the image of  $a \otimes b$  under (46); we will write the variable corresponding to  $a$  (resp.  $b$ ) under  $\theta$  as  $z$  (resp.  $t$ ). We then perform insertion to the variable  $t$ , calling the new variable  $u$ , and we shall set  $t = 0$ , and take the linear term with respect to  $u$ . Taking the path through the upper right corner of (46), we obtain

$$(47) \quad a(z)L_{-1}b,$$

while taking the path through the lower left corner, we obtain two terms corresponding to  $u$ -linear terms of the expansion

$$z^n = \sum_{i \geq 0} \binom{n}{i} (z-u)^{n-i} u^i,$$

namely coming from  $-n(z-u)^{n-1}$  and  $(z-u)^n u$ . In this order, this corresponds to the first and second term of the sum

$$(48) \quad -a'(z)b + L_{-1}a(z)b.$$

The equality between (47) and (48) is (26). All the other axioms are obvious, which completes the proof of the statement of the Theorem for general and bounded VA-algebras.

The statement for the  $k$ -connective follows easily: clearly, if we have a  $k$ -connective VA algebra  $X$ , then by applying

$$\theta : X \rightarrow X \otimes \mathcal{C}(1)$$

and substituting 1 for the variable  $z$  (i.e. applying the augmentation  $\mathcal{C}(1) \rightarrow \mathcal{C}(0)$ ), we get the unit, but when starting with element of degree  $< k$ , the corresponding part of the  $\mathbb{Z}$ -sorted operad  $\mathcal{C}_k$  is 0, so the identity is 0 on those elements, proving that we have a  $k$ -connective vertex algebra.

On the other hand, starting with a  $k$ -connected vertex algebra, using the  $\mathcal{C}$ -co-algebra structure which we already proved, co-associativity and Theorem 10, we see that (36) specializes to a map

$$V_{k_0} \otimes V_{k_1} \otimes \dots \otimes V_{k_n} \rightarrow \mathcal{C}_k(k_\infty, k_0, \dots, k_n) \otimes V_{k_\infty},$$

as required. □

**Remark:** If we want to work in the algebraic category (over a field of characteristic 0), the convergence arguments in the beginning of the proof need to be replaced with a discussion of “order of expansion”, as treated, say, in [8].

#### 4. CONSTRUCTIONS OF VERTEX ALGEBRAS FROM GENERATORS AND DEFINING RELATIONS

By Theorem 13 and the remarks following Lemma 8, the category of  $k$ -connective vertex algebras is equivalent to a category of algebras over an operad, and the equivalence commutes with the forgetful map, so we can speak of free  $k$ -connective vertex algebras on a graded vector space and by the remarks preceding Lemma 34, also of  $k$ -connective vertex algebras defined by means of generators and defining relations. Note that although we haven’t identified an exact vector space basis of a free  $k$ -connective vertex algebra  $C_k X$  on a graded vector space  $X$ , by (95), and Theorem (10), we have a reasonably explicit presentation of  $C_k X$  in terms of generators and defining relations in the category of graded vector spaces. The main purpose of this section is to give the presentations of certain well known vertex algebras by generators and defining relations.

**Remark 14.** Clearly, the operadic notation is awkward for the purposes of practical applications. From Lemma 2, (or alternately simply Definition 11) we see that every element in a ( $k$ -connective) vertex algebra generated by certain elements can be written by finite words using the binary operation

$$(49) \quad a(n)b, \quad n \in \mathbb{Z}$$

and unary operation

$$(50) \quad L_{-1}a$$

in the indeterminates  $a, b$ . It is appealing from this point of view to rewrite (49) as

$$(51) \quad [a, b]_n$$

and (50) as

$$(52) \quad a'.$$

The following Lemma then suggests that vertex algebras can be understood as a sort of “deformation” of the notion of Lie algebras. Indeed, the Lemma for example clearly exhibits the known fact that for a vertex algebra  $V$ ,  $V/L_{-1}V$  is a Lie algebra with respect to the operation  $[\cdot, \cdot]_0$ . Despite of the appeal of the notation of Lemma 15, however, the standard notation (49), (50), and especially the field notation, is often preferable in obtaining intuition from mathematical physics.

**Lemma 15.** *In a vertex algebra, we have*

$$(53) \quad [a', b]_n = n[a, b]_{n-1},$$

$$(54) \quad [a, b]'_n = [a', b]_n + [a, b']_n,$$

$$(55) \quad [b, a]_m = (-1)^{m+1} \sum_{j \geq 0} \frac{1}{j!} [a, b]_{j+m}^{(j)},$$

$$(56) \quad [a, [b, c]_n]_m - [b, [a, c]_m]_n = \sum_{i \geq 0} \binom{m}{i} [[a, b]_i, c]_{m+n-i}.$$

**Proof:** (56) is an immediate rewrite of (32). (53), (54) are usual properties of the shift in vertex algebras, proved for example in [13]. (55) is a consequence of those properties and (56).  $\square$

In figuring out the structure of a vertex algebra in terms of generators and defining relations, it is usually key to understand first the relations among the operations (49) for  $n \geq 0$ . This corresponds, in physical language, to figuring out the operator product expansion (OPE) of the generators. In many cases, the Existence theorem (see [6], Kac [13], Theorem 4.5) then determines explicitly the structure of the vertex algebra in question.

More precisely, we have:

**Proposition 16.** *Consider an  $m$ -connected vertex algebra given by an ordered set of generators  $B$  and relations equating every (49) for  $a, b \in B$ ,  $n \geq 0$ ,  $a \leq b$  with a linear combination of derivatives of elements of  $B$ , then by Proposition 56,*

$$(57) \quad b_1(n_1) \dots b_k(n_k)(1)$$

*with  $b_i \in B$ ,  $0 > n_1 \geq \dots \geq n_k$ ,  $n_i = n_{i+1} \Rightarrow b_i \geq b_{i+1}$  generates  $V$  as a vector space.*

$\square$

**Example 17.** [7] Let  $L$  be any Lie algebra with a 2-cocycle represented as an invariant symmetric bilinear form  $\langle \cdot, \cdot \rangle$ . Then the Lie vertex algebra is the 0-connective vertex algebra  $VL$  with generators  $L$  in dimension 1, and relations

$$(58) \quad [a, b]_0 = [a, b],$$

$$(59) \quad [a, b]_1 = \langle a, b \rangle \cdot 1.$$

Then for dimensional reasons, no further operations (49) with  $n \geq 0$  are possible, and hence by (56), if we denote by  $B$  an ordered basis of  $L$ , then (57) generates  $VL$ , and by the existence theorem (Theorem 4.5 of [13],) (57) is in fact a vector space basis of  $VL$ .

**Example 18.** The Virasoro vertex algebra in the category of 0-connective vertex algebras has one generator  $L$  of dimension 2 and relations

$$(60) \quad [L, L]_0 = L', \quad [L, L]_1 = 2L, \quad [L, L]_2 = 0, \quad [L, L]_3 = (c/2) \cdot 1$$

where  $c$  is the central charge. Again, this closes the OPE algebra, and therefore again a vector space set of generators of the Virasoro vertex algebra is (57) where  $B = \{L\}$ . Again, this is a basis (cf. Frenkel [7]) by the Existence theorem.

Sometimes, the OPE is not sufficient for determining the structure of the vertex algebra completely when there are relations between the higher derivatives of the generators.

**Example 19.** Let  $L$  be an even positive-definite lattice with  $\mathbb{Z}/2$ -valued bilinear form  $B$  such that for all  $x \in L$ ,

$$(61) \quad b(x, x) = \langle x, x \rangle / 2.$$

Then the lattice vertex algebra  $V_L$  is

$$(62) \quad VL_{\mathbb{C}} \otimes \mathbb{C}[L]$$

where  $L_{\mathbb{C}} = L \otimes \mathbb{C}$  is the trivial Lie algebra with symmetric bilinear form coming from  $L$  and  $\mathbb{C}[L]$  is the group algebra (we denote the canonical basis elements of  $\mathbb{C}[L]$  by  $(\lambda)$ ,  $\lambda \in L$ ). The  $VL_{\mathbb{C}}$  factor is a vertex subalgebra whose structure is specified in Example 17. More generally, we interpret a product  $x \otimes (\lambda)$  in (62) as  $x(-1)\lambda$ . Then in this notation, we have

$$(63) \quad a(0)(\lambda) = \langle a, \lambda \rangle (\lambda), \quad a(n)(\lambda) = 0, \quad n > 0, \quad a \in L_{\mathbb{C}},$$

and for  $x \in VL_{\mathbb{C}}$ ,

$$(64) \quad (\lambda)(z)x(-1)(\mu) = (-1)^{b(\lambda, \mu)} (\exp(\partial_z^{-1} \lambda(z)) : x)(-1)(\lambda + \mu)$$

where the antiderivative is interpreted in the usual way, cf. [8].

We can therefore take the generators of  $V_L$  to be

$$(65) \quad \lambda_{\mathbb{C}}, (\lambda),$$

where  $\lambda \in L$ . The formulas (63), (64) certainly determine the OPE of the generators (65). More concretely, recall the convention that we write  $a_+(z)$  resp.  $a_-(z)$  for the sum of terms with non-negative resp. negative power of  $z$  in  $a(z)$ , then the corresponding *normal order* expression is

$$: a(z)b(t) := a_+(z)b(t) + b(t)a_-(z).$$

Then we have

$$(66) \quad (\lambda)(z)(\mu)(t) =: (\lambda(z)(\mu)(t) : +(z-t)^{\langle \lambda, \mu \rangle} (\lambda + \mu)(t),$$

$$(67) \quad \mu_{\mathbb{C}}(z)(\lambda)(t) =: \mu_{\mathbb{C}}(z)(\lambda)(t) : +\langle \lambda, \mu \rangle (z-t)^{-1} (\lambda)(t),$$

$$(68) \quad \mu_{\mathbb{C}}(z)\lambda_{\mathbb{C}}(t) =: \mu_{\mathbb{C}}(z)\lambda_{\mathbb{C}}(t) : +\langle \lambda, \mu \rangle (z-t)^{-2}.$$

When considering operations of the form (51) with  $n \geq 0$ , then the normal order expression vanishes. Therefore, clearly the relations we get from (65) satisfy the assumptions of Proposition 16 if we choose the order in such a way that

$$\lambda_{\mathbb{C}} < (\mu)$$

for any  $\lambda, \mu \in L$ .

Even then, however, we see that there are more expressions (57) than a basis of (62), since (62) allows only one label generator, and no derivatives. We can remedy the situation by including the relation

$$(0) = 1$$

and a relation following from (64) which expresses any  $\lambda(n)\mu$ ,  $n < 0$  as a product of terms of the form  $\mu_{\mathbb{C}}(m)$ ,  $m < 0$  from the left with  $(\lambda + \mu)$ .

Nevertheless, in this case, there is an extremely clever solution given by the following

**Proposition 20.** [17] *Let  $\mathbb{P}^{\infty}$  be a basis of the lattice  $L$ , and assume that  $b(x, x) = 1$  for  $x \in \mathbb{P}^{\infty}$ . Then the in the category of 0-connective vertex algebras,  $V_L$  can be presented in generators*

$$(\lambda), \lambda \in \pm\mathbb{P}^{\infty},$$

and relations

$$\begin{aligned} (\lambda)(n)(\mu) &= 0 \text{ when } n < -\langle \lambda, \mu \rangle, \\ (\lambda)(\|\lambda\|^2)(-\lambda) &= 1. \end{aligned}$$

**Proof:** Roitman [17], Section 10 proves that the canonical morphism

$$(69) \quad \psi : V \rightarrow V_L$$

from the vertex algebra  $V$  given by the given generators and defining relations in his category to  $V_L$  is an isomorphism. Therefore,  $V$  is in particular 0-connective. However, since the 0-connectivity restriction can be viewed as additional relations in Roitman's category, (69) factors through a morphism

$$(70) \quad \tau : V \rightarrow V_0$$

where  $V_0$  is the 0-connected vertex algebra with the given generators and defining relations. Since, in the present case,  $\psi$  is iso, so is  $\tau$ .  $\square$

One canonical way of always selecting relations is as follows: let  $V$  be a  $k$ -connective vertex algebra,  $k \leq 0$ . Consider then the relations

$$(71) \quad \begin{aligned} a &= 0 \text{ for } a \in V \text{ whenever } |b_1(n_1)\dots b_k(n_k)a| \leq 0 \text{ implies} \\ b_1(n_1)\dots b_k(n_k)a &= 0 \text{ for } b_1, \dots, b_k \in V, n_1, \dots, n_k \in \mathbb{Z}. \end{aligned}$$

**Lemma 21.** *As a vector space, the quotient of  $V$  by the relations (71) is isomorphic to  $V/I$  where*

$$(72) \quad I = (L_{-1})^m(b_1(n_1)\dots b_k(n_k)a) \text{ where } b_i \in V, n_i \in \mathbb{Z} \text{ and } a \text{ is as in (71), } m \geq 0.$$

*In particular,*

$$(73) \quad V/I \neq 0.$$

**Proof:** By Lemma 34, the map from  $V$  to the quotient by the relations (71) is onto. Further, by its definition,  $I$  is clearly contained in the kernel. Thus, by universality it suffices to show that  $V/I$  is indeed a vertex algebra, or that  $I$  is an ideal in the category of  $k$ -connective vertex algebras considered as universal algebras. This means that  $I$  must be stable under all operations where precisely one entry is in  $I$  and the other entries are in  $V$ . Clearly,  $I$  is stable under the derivative. Additionally, clearly  $a \in V$  and  $u \in I$  together imply  $a(n)u \in I$ , by (53) and (54). It then follows that also  $u(n)a \in I$  by (55). The assertion (73) now clearly follows from the condition on  $a$  in (71).  $\square$

We shall refer to the ideal  $I$  from Lemma 21 as the *maximal positive ideal*. The quotients of Lie vertex algebras (resp. Virasoro vertex algebras) by the maximal positive ideal  $I$ , when  $I \neq 0$ , are referred to as WZW models (resp. minimal models). (There are variants of all these notions involving supersymmetry, but we won't discuss them here.) One can also prove that the lattice vertex algebra can be characterize in this way:

**Proposition 22.** *Let  $V$  be the vertex algebra with generators (65) and relations for  $a(n)b$  where  $n \leq 0$ , and  $a, b$  are generators, given by the corresponding summands of (64), and let  $I$  be the maximal positive ideal in  $V$ . Then  $V/I$  is the lattice algebra corresponding to  $L$ .*

**Proof:** The key point is to show that the maximal positive ideal of  $V_L$  is 0; then there exists a map of 0-connective vertex algebras

$$V/I \rightarrow V_L$$

which in turn must be an iso, because its kernel would be contained in the maximal positive ideal of  $V/I$ , which must be 0.

To show that the maximal positive ideal of  $V_L$  is 0, we invoke the well known fact [8] that  $V_L$  is semisimple and its irreducible modules correspond to elements of  $L'/L$  (where  $L'$  is the dual lattice). For any semisimple vertex algebra, the maximal positive ideal is 0 (for example by Zhu [20]).  $\square$

**Example 23.** The Moonshine module  $V^\natural$ . Let  $B$  be Griess' commutative non-associative algebra [10]. Then  $B$  is the weight 2 summand of the Moonshine module [8], and the operation in  $B$  is  $[\cdot, \cdot]_1$ . The operation  $[\cdot, \cdot]_2$  is 0 and the operation  $[\cdot, \cdot]_3$  is the pairing invariant under the action of the Monster. We can take the free 0-connected vertex algebra on the generator space  $B$  with these relations. Let  $I$  be the maximal positive ideal in  $V$ . Then one has

$$(74) \quad V/I \cong V^\natural.$$

The reason is similar as in the lattice case:  $V^{\natural}$  is semisimple and has only one irreducible module, so its maximal positive ideal is 0. This induces a map from the left hand side to the right hand side of (74). The map is onto because  $V^{\natural}$  is generated by its elements of weight 2 (see [8]). Any kernel would again be contained in the maximal positive ideal of  $V/I$ , which is 0.

Determining completely the OPE of the Moonshine module in the sense of Proposition 16 is an interesting problem. This is because we have the operation

$$(75) \quad [?, ?]_0.$$

In fact, as we remarked above, for any vertex algebra  $V$ ,  $V/L_{-1}V$  forms a Lie algebra under the operation (75). In the case of  $V^{\natural}$ , it is an interesting problem to determine the structure of this Lie algebra  $L$ : the weight 3 part of  $V^{\natural}$  decomposes as a representation of the Monster into dimensions

$$21493760 = 21296876 + 196883 + 1.$$

The second two terms are the image of  $L_{-1}$ . Thus, we see that  $L_3$  is the irreducible representation of dimension 21493760 and  $L_2$  is the irreducible representation of dimension 196883. In particular, the Lie bracket exhibits the 21493760-dimensional representation as a quotient of the second exterior power of the 196883-dimensional representation.

If we knew the complete structure of  $L$ , then we would know the OPE in the sense of Proposition 16; the operations  $[?, ?]_1$  and (75) satisfy the relations

$$\begin{aligned} [a, [b, c]_1]_0 - [b, [a, c]_0]_1 &= [[a, b]_0, c]_1, \\ [a, [b, c]_0]_1 - [b, [a, c]_1]_0 &= [[a, b]_0, c]_1 + [[a, b]_1, c]_0, \end{aligned}$$

which are special cases of (56).

An easier approach to finding a complete representation of the 0-connective vertex algebra  $V^{\natural}$  by generators and defining relations seems to be from the construction of Frenkel, Lepowsky and Meurman [8]. They constructed  $V^{\natural}$  as the sum of the vertex algebra  $V_L^{\theta}$  where  $L$  is the Leech lattice and  $\theta$  is the involution  $\lambda \mapsto -\lambda$  and  $V_{L,T}^{\theta}$  where  $V_{L,T}$  is the twisted module of  $V_L$  with respect to the involution  $\theta$ . Let us focus on the untwisted sector. We can choose generators

$$(76) \quad \mu_{\mathbb{C}}\nu_{\mathbb{C}},$$

$$(77) \quad (\lambda) + (-\lambda),$$

$$(78) \quad \mu_{\mathbb{C}}(\lambda) - \mu_{\mathbb{C}}(-\lambda)$$

where  $\lambda, \mu, \nu \in L$ . These generators do not close under the OPE operations of Proposition 16. However, we can express a general element of  $V_L^{\theta}$  as a product

$$(79) \quad b_1(n_1) \dots b_k(n_k) a$$

where  $n_i < 0$ ,  $b_i$  are of the form (76) and  $a$  is of the form (77) or (78). Using (68), we can see that the elements (76) close under the OPE, and hence the elements  $b_i$  in (79) can be reordered using their OPE relations. Using (67) and (68), one can also write relations equating the appropriate elements

$$(80) \quad b(n)a, \quad n < 0$$

where  $b$  is of the form (76) and  $a$  is of the form (77) (corresponding to reordering the individual  $\mu_{\mathbb{C}}$  factors in (68)). Finally, using (66) - (68) again, we can describe relations for applying (49) with  $a$  of the form (77) or (78) where  $\|\lambda\| = 2$  to

(76)-(78) and yielding results of the form (79). We omit the details. Putting all these relations together, we obtain a presentation of  $V_L^\theta$  in terms of generators and defining relations. Adding relations describing products (49) where  $a, b$  are weight 2 elements of  $V_T^\theta$ , we could write down an explicit presentation of  $V^\natural$  in terms of generators and defining relations.

## 5. APPENDIX: SOME FACTS ON CATEGORIES AND OPERADS

Universal algebras with underlying structure of a vector space where the additional operations are distributive can be often axiomatized as algebras over an operad. Co-operads become useful when certain finiteness conditions fail. Because of variations in the definitions, we recall here the definition of an operad, co-operad, and universal algebra over them. We will work in the underlying category  $Vect$  of  $\mathbb{C}$ -vector spaces and homomorphisms. We will also consider the category  $\mathbb{Z} - Vect$  of  $\mathbb{Z}$ -graded  $\mathbb{C}$ -vector spaces and homomorphisms preserving degree.

**Definition 24.** An *operad*  $\mathcal{C}$  in  $Vect$  is a sequence of objects  $\mathcal{C}(n)$ ,  $n = 0, 1, 2, \dots$  with right  $\Sigma_n$ -action, and morphisms

$$(81) \quad \gamma : \mathcal{C}(n) \otimes \mathcal{C}(m_1) \otimes \dots \otimes \mathcal{C}(m_n) \rightarrow \mathcal{C}(m_1 + \dots + m_n)$$

such that we have equivariance, stating that for permutations  $\tau \in \Sigma_n$ ,  $\sigma_i \in \Sigma(m_i)$ ,

$$(82) \quad \begin{array}{ccc} \mathcal{C}(n) \otimes \mathcal{C}(m_1) \otimes \dots \otimes \mathcal{C}(m_n) & & \\ \text{Id} \otimes \sigma_1 \otimes \dots \otimes \sigma_n \downarrow & \searrow^{\tau \wr \sigma_1, \dots, \sigma_n} & \\ \mathcal{C}(n) \otimes \mathcal{C}(m_1) \otimes \dots \otimes \mathcal{C}(m_n) & & \\ \tau \otimes T \downarrow & & \\ \mathcal{C}(n) \otimes \mathcal{C}(m_{\tau(1)}) \otimes \dots \otimes \mathcal{C}(m_{\tau(n)}) & \xrightarrow{\gamma} & \mathcal{C}(m_1 + \dots + m_n). \end{array}$$

( $\wr$  denotes the wreath product of permutations and  $T$  is the switch) and associativity, which states that

$$(83) \quad \begin{array}{ccc} \mathcal{C}(n) \otimes M \otimes L_1 \otimes \dots \otimes L_n & & \\ \text{Id} \otimes \gamma \otimes \dots \otimes \gamma \downarrow & \searrow^{\gamma \otimes \text{Id} \otimes \dots \otimes \text{Id}} & \\ \mathcal{C}(n) \otimes \mathcal{C}(\ell_1) \otimes \dots \otimes \mathcal{C}(\ell_n) & \xrightarrow{\gamma} & \mathcal{C}(\ell_1 + \dots + \ell_n) \\ & & \downarrow \gamma \\ & & \mathcal{C}(m_1 + \dots + m_n) \otimes L_1 \otimes \dots \otimes L_n \end{array}$$

where  $\ell_i = \ell_{i1} + \dots + \ell_{ik_i}$ ,

$$M = \mathcal{C}(m_1) \otimes \mathcal{C}(m_2) \otimes \dots \otimes \mathcal{C}(m_n),$$

$$L_i = \mathcal{C}(\ell_{i1}) \otimes \dots \otimes \mathcal{C}(\ell_{ik_i}).$$

Since (82), (83) are expressed in terms of diagrams, we may therefore define a *co-operad* as a sequence of objects  $\mathcal{C}(n)$ ,  $n = 0, 1, 2, \dots$ , a left  $\Sigma_n$  action on  $\mathcal{C}(n)$  and morphisms

$$(84) \quad \gamma : \mathcal{C}(m_1 + \dots + m_n) \rightarrow \mathcal{C}(n) \otimes \mathcal{C}(m_1) \otimes \dots \otimes \mathcal{C}(m_n)$$

such that diagrams dual to (82) and (83) commute. Definitions of operads and co-operads over  $\mathbb{Z} - Vect$  are the same, except we work in the category  $\mathbb{Z} - Vect$ . Specifically, a  $\mathbb{Z}$ -graded co-operad is a collection of spaces  $\mathcal{C}(n)_k$ ,  $k \in \mathbb{Z}$  and for  $k = k_1 + \dots + k_n$ , maps

$$(85) \quad \gamma : \mathcal{C}(m_1 + \dots + m_n)_k \rightarrow \mathcal{C}(m_1)_{k_1} \otimes \dots \otimes \mathcal{C}(m_n)_{k_n}$$

which satisfy the obvious analogue of the axioms (82), (83).

**Remark 25.** The operations (81) are symmetrical, but ‘wasteful’ in the sense that they can be generated by less general operations, called *insertions*, and the notion of an operad can be equivalently formulated in terms of these operations. An insertion is an operation

$$(86) \quad \phi : \mathcal{C}(m+1) \otimes \mathcal{C}(n-m) \rightarrow \mathcal{C}(n),$$

and is defined by

$$\phi(x, y) = \gamma(x, 1, \dots, 1, y).$$

The axioms are equivariance (the restriction of general equivariance to the cases which apply to insertions), associativity of insertions, and a commutativity of insertions into two different variables (insertion into a different variable than the last is defined using the equivariance). This clearly extends to the other structures defined in Definition 24. Both formulations are easily seen to be equivalent.

**Remark 26.** First note that there is a stronger variant of axiom (85) in the case of graded co-operads, as (85) allows an ‘infinite sum’ when one varies the degrees  $k_1, \dots, k_n$  with given sum  $k$  on the right hand side. We could require that all but finitely many terms in this sum are 0. This stronger axiom is *not* what we mean in this paper.

Next, note that  $\mathbb{Z}$ -graded operads and co-operads model situations when we have operations which shift total degree by a given number, but the operations which apply to each tuple of degrees are exactly the same. When we want to capture the situation where available operations depend on the input degrees, we must introduce the notion of  *$\mathbb{Z}$ -sorted operad*. This is a system of vector spaces  $V(m_0, m_1, \dots, m_n)$ ,  $m_i \in \mathbb{Z}$  where the right action of  $\sigma \in \Sigma_n$  has

$$\sigma : V(m_0, m_1, \dots, m_n) \rightarrow V(m_0, m_{\sigma(1)}, \dots, m_{\sigma(n)})$$

and composition is defined as

$$\gamma : V(m_0, m_1, \dots, m_n) \otimes V(m_1, m_{11}, \dots, m_{1,k_1}) \otimes \dots \otimes V(m_n, m_{n1}, \dots, m_{n,k_n}) \rightarrow V(m_0, m_{11}, \dots, m_{nk_n}).$$

The diagram axioms (82), (83) are the same. Because the axioms are again in shapes of diagrams, we have a dual notion of  $\mathbb{Z}$ -sorted co-operads. In fact, here, one may replace  $\mathbb{Z}$  by any set  $I$ , and define  *$I$ -sorted operads* (or co-operads) in the same fashion.

**Proposition 27.** *The forgetful functor from the category of operads (and the obvious homomorphisms - maps preserving the operations) to vector spaces is a right adjoint (similarly for the forgetful functor from graded and  $\mathbb{Z}$ -sorted operads to the*

category of  $\mathbb{N} \times \mathbb{Z}$ -graded vector spaces and vector spaces graded by  $\coprod_{n \geq 0} \mathbb{Z}^n$ . The forgetful functor from the category of co-operads (resp. graded co-operads, resp.  $\mathbb{Z}$ -sorted co-operads) to  $\mathbb{N}$ -graded vector spaces (resp. the category of  $\mathbb{N} \times \mathbb{Z}$ -graded vector resp. vector spaces graded by  $\coprod_{n \geq 0} \mathbb{Z}^n$ ) is a left adjoint.

We shall postpone the proof because it is an example of an even more general principle.

**Definition 28.** Let  $\mathcal{C}$  be an  $I$ -sorted operad. A  $\mathcal{C}$ -algebra is a system of vector spaces  $X_i$ ,  $i \in I$ , and a system of maps

$$(87) \quad \theta : \mathcal{C}(i_0, i_1, \dots, i_n) \otimes X_{i_1} \otimes \dots \otimes X_{i_n} \rightarrow X_{i_0}$$

satisfying

$$(88) \quad (\theta\sigma)(?, \dots?) = \theta(\sigma(?, \dots?))$$

where  $\theta \in \Sigma_n$  and  $\sigma$  acts on tuples by permutation, and

$$(89) \quad \theta(\gamma(?, ?, \dots?), ?, \dots?) = \theta(?, \theta(?, ?, \dots?), \dots, \theta(?, ?, \dots?))$$

when applicable.

**Remark 29.** In the previous definition, one may also define an endomorphism operad and coendomorphism operad of  $X$  by

$$(90) \quad \text{End}(X)(i_0, i_1, \dots, i_n) = \text{Hom}(X_{i_1} \otimes \dots \otimes X_{i_n}, X_{i_0}),$$

$$(91) \quad \text{Coend}(X)(i_0, i_1, \dots, i_n) = \text{Hom}(X_{i_0}, X_{i_1} \otimes \dots \otimes X_{i_n}).$$

Then a  $\mathcal{C}$ -algebra is the same thing as a homomorphism of  $I$ -sorted operads

$$(92) \quad \mathcal{C} \rightarrow \text{End}(X).$$

It is therefore natural to define a coalgebra  $X$  over an  $I$ -sorted operad  $\mathcal{C}$  to be a morphism of operads

$$(93) \quad \mathcal{C} \rightarrow \text{Coend}(X).$$

This, of course can also be written in terms of maps

$$(94) \quad \theta : \mathcal{C}(i_0, i_1, \dots, i_n) \otimes X_{i_0} \rightarrow X_{i_1} \otimes \dots \otimes X_{i_n}$$

and conditions analogous to (88), (89).

We can also define algebras resp. coalgebras over a co-operad by dualizing (87), (88), (89) resp. (94) and the corresponding conditions. However, in for these dual notions, we do not know of an analogue of the descriptions (92), (93).

**Proposition 30.** For every set  $I$ , there exists an  $\coprod_{n \geq 0} I^{n+1}$ -sorted operad  $Q_I$  such that the category of  $I$ -sorted operads (resp. co-operads) is canonically equivalent to the category of  $Q_I$ -algebras (resp.  $Q_I$ -co-algebras).

□

Proposition 27 now follows from the following result.

**Proposition 31.** Let  $\mathcal{C}$  be an  $I$ -sorted operad. Then the forgetful functor from the category of  $\mathcal{C}$ -algebras (resp.  $\mathcal{C}$ -coalgebras) to  $I$ -graded vector spaces is a right (resp. left) adjoint.

**Proof:** The statement about  $\mathcal{C}$ -algebras is classical. In effect, the left adjoint is

$$(95) \quad CX_i = \bigoplus_{n \geq 0} \left( \bigoplus_{(i, i_1, \dots, i_n) \in I^{n+1}} \mathcal{C}(i, i_1, \dots, i_n) \otimes X_{i_1} \otimes \dots \otimes X_{i_n} \right) / \Sigma_n,$$

and in fact the category of  $\mathcal{C}$ -algebras is equivalent to the category of algebras over the monad  $C$ . (For the definition of monads and their algebras, see [14].)

The statement about  $\mathcal{C}$ -coalgebras follows from the fact that the forgetful functor from  $\mathcal{C}$ -coalgebras to  $I$ -graded vector spaces obviously preserves coproducts and coequalizers, and one can use the adjoint functor theorem to show it must have a right adjoint.  $\square$

**Remark 32.** It is not true that the right adjoint of the forgetful functor from  $\mathcal{C}$ -coalgebras to  $I$ -graded vector spaces would be given by a dual of (96). The dual is

$$(96) \quad PX = \prod_{n \geq 0} \left( \prod_{(i, i_1, \dots, i_n) \in I^n} \text{Hom}(\mathcal{C}(i, i_1, \dots, i_n), X_{i_1} \otimes \dots \otimes X_{i_n})^{\Sigma_n} \right),$$

but (96) fails to be a comonad. The problem is that tensor product does not distribute under the product, so when one writes the formula for a map

$$PX \rightarrow PPX$$

dual to the classical monad structure on  $C$ , the image actually will be a vector space containing but not equal to  $PPX$ . The actual comonad is the intersection  $P'X$  of the inverse images of all such maps  $X \rightarrow P^n X$ .

**Remark 33.** Another example of failure of dualization is the fact that the forgetful functor from the category of algebras over an  $I$ -sorted cooperad  $\mathcal{C}$  to the category of  $I$ -graded vector spaces does *not* in general create products. Note that the structure map is

$$(97) \quad X_{i_1} \otimes \dots \otimes X_{i_n} \rightarrow \mathcal{C}(i, i_1, \dots, i_n) \otimes X_i.$$

When we take a product of  $\mathcal{C}$ -coalgebras  $X(j)$ , then the image of the structure map (97) for  $\prod X(j)$  ends up in

$$\prod \mathcal{C}(i, i_1, \dots, i_n) \otimes X(j)_i,$$

while  $\mathcal{C}$ -coalgebra structure requires it to be in the subspace

$$\mathcal{C}(i, i_1, \dots, i_n) \otimes \prod X(j)_i.$$

However, note that when all of the spaces  $\mathcal{C}(i, i_1, \dots, i_n)$  are finite-dimensional, then the collection of the dual spaces  $\mathcal{C}(i, i_1, \dots, i_n)^\vee$  forms an  $I$ -sorted operad  $\mathcal{C}^\vee$ , and the category of  $\mathcal{C}$ -algebras is equivalent to the category of  $\mathcal{C}^\vee$ -algebras, which, as remarked above, is canonically equivalent to the category of monads over the category of  $I$ -graded vector spaces. In particular, in that case, the forgetful functor does create products.

Let  $\mathcal{C}$  be an  $I$ -sorted operad. There is one more construction we need to cover, namely quotient  $\mathcal{C}$ -algebras, and the dual construction for  $\mathcal{C}$ -coalgebras. The most general version of the story is this: Let

$$(98) \quad U : C \rightarrow D$$

be a functor with left adjoint  $L$ , and let  $X$  be an object of  $C$ . Then we can consider the category  $X/C$  of arrows  $X \rightarrow ?$ , where  $?$  is an object or morphism of  $C$ . The forgetful functor

$$X/C \rightarrow C$$

creates limits. Then  $U$  induces a functor

$$(99) \quad X/U : X/C \rightarrow UX/D,$$

which therefore preserves limits, and by the adjoint functor theorem has a left adjoint  $X/L$ . Similarly, when (98) has a right adjoint  $R$ , and  $X$  is an object of  $C$ , we may consider the category  $C/X$  of arrows  $? \rightarrow X$ , and the forgetful functor

$$(100) \quad U/X : C/X \rightarrow D/UX$$

preserves colimits, and hence has a right adjoint  $R/X$  by the adjoint functor theorem. Note that this construction applies to the cases of  $\mathcal{C}$ -algebras and  $\mathcal{C}$ -coalgebras by Proposition 31.

**Lemma 34.** *When  $D$  is the category of  $I$ -graded vector spaces and  $C$  is the category of algebras (resp. coalgebras) over an  $I$ -sorted operad,  $U$  is the forgetful functor (see (98)) and  $L$  (resp.  $R$ ) is the left (resp. right) adjoint, then the functor  $X/L$  (resp.  $R/X$ ) preserves epimorphisms (resp. monomorphisms).*

**Proof:** The key point is that for a morphism

$$(101) \quad f : A \rightarrow B$$

of  $\mathcal{C}$ -algebras (resp.  $\mathcal{C}$ -coalgebras), the image of  $f$  is a  $\mathcal{C}$ -algebra (resp.  $\mathcal{C}$ -coalgebra). This factorizes any morphism (101) as

$$(102) \quad f = gh \text{ where } g \text{ is mono and } h \text{ is epi.}$$

But if we take an epimorphism (resp. monomorphism)  $\phi$  in  $UX/D$  (resp.  $D/UX$ ), and consider  $f = X/L(\phi)$ , then factoring  $f$  as in (102),  $h$  (resp.  $g$ ) enjoys the same universal property as  $f$ , and hence must be isomorphic to  $f$ .  $\square$

**Remark 35.** In the case of  $\mathcal{C}$ -algebras, we can interpret  $X/L(\phi)$  where  $\phi$  is an epimorphism of  $I$ -graded vector spaces as the quotient of  $X$  by the ideal generated by  $\text{Ker}(\phi)$ . In the case of  $\mathcal{C}$ -coalgebras, the interpretation is dual.

(Note that operads in our sense describe algebras with underlying vector space structure and multilinear operations with multilinear relations. In this setting, taking quotients with respect to a congruence reduces to taking quotients by an ideal.)

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