STRINGY BUNDLES AND INFINITE LOOP SPACE THEORY

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1. introduction

In this note, we describe a new a candidate for an elliptic cohomology-type spectrum based on conformal field theory. The new model is an improvement of the constructions of [4]. This model relies on a new infinite loop space machine deviced by Gomez [3]. The present model is interesting because it is, in some sense, a compromise between the models proposed in [4] and [6].

To compare the definition of a spectrum $\mathfrak{E}$ in this note with the construction of [4], one major difference is the use of the infinite loop space machine [3]. In [4], no infinite loop space machine was in sight, and because of that, an "Ersatz" construction of taking the suspension spectrum and inverting a suitable element was considered. Also, we relate the notions involved to more familiar mathematical constructions (conformal field theories on a given SPCMC, instead of just defining stringy bundles ad hoc). Another major difference is that in the present definition, we give up the "manifest modularity" feature, i.e. we do not consider translation-equivariant stringy bundles over an elliptic curve E, but only stringy bundles over $\mathbb{C}$ with compact support. It should be pointed out that there exists a version of our construction which does reproduce the "manifestly modular" approach, although there are certain subtleties involving equivariant stacks, namely, one must vary the sites involved to reproduce the continuity of the action.

The reason we do not insist on manifest modularity in our definition of choice is that all the calculational observations made in [4] can, in fact, be reproduced in the new setting. The three features reproduced are the character map

$$\mathfrak{E} \to K[[q]], \tag{1}$$

a model of the 4-dimensional characteristic integral cohomology class

$$BE_8 \to K(\mathbb{Z}, 4), \tag{2}$$

and a map

$$\widetilde{BE_8} \to \mathfrak{E} \tag{3}$$

where $\widetilde{BE_8}$ is the homotopy fiber of the map (2). The calculational evidence found in [4] can be summarized as follows: one might naively want to simply define our spectrum by applying the machine [3] to the category of CFT's with modular functor over $C(\mathbb{C}_2)$ fibered over the category of modular functors over $C(\mathbb{C}_2)$. Indeed, this allows a natural map (1) and also (3), but one finds that the image of such map on homotopy does not consist of modular forms. In particular, since we may work rationally, we can consider the images of duals of the primitive elements of H^*BE_8, which are rational homotopy classes of BE_8. One finds that the image of the 4-dimensional homotopy class is not a modular form, whereas the images of the higher dimensional classes are modular forms.

This suggests that there should be a class

$$BAut(H) \to K(\mathbb{Z}, 4), \tag{4}$$

where $Aut(H)$ is the automorphism group of a CFT (H) and the homotopy classes which are modular forms should be in its fiber. Unfortunately, we know of no construction of a natural class of the form (4), although it is interesting to note that Stolz and Teichner [6], Section 5.4, arrive at a geometric model of the class (2) by using structures obtaining by their conjectured extension of CFT. This is quite provocative, and we think there must be a connection. In our setting, however, the only way such class seems to appear is by considering "bundles of CFT's" over a 2-dimensional complex manifold X (such as E or $\mathbb{C}$ with compact supports). The "bundles of CFT's" in addition means CFT's on some SPCMC of worldsheets embedded in X, which leads to the setup we consider here.

2. The definition

Let us recall briefly what a SPCMC is. It is a concept which axiomatizes in the most complete way known to us the structure present on the set of all worldsheets, i.e. compact Riemann surfaces with parametrized boundary. (So as not to worry about set theory, all of the objects are bounded in cardinality, so let they all be subsets of some fixed universe which is a set.)

'S' stands for 'stack', and 'P' for 'pseudo', so we should first consider CMC's, which stands for 'commutative monoid with cancellation'. With some terminological variations and errors, this is defined in [4, 1], with the corrections discussed in [2]. A commutative monoid with cancellation consists of an underlying commutative monoid I, and for $i, j \in I$, a set $X_{i,j}$. The operations are

$$0 \in X_{0,0}, \tag{5}$$

$$+ : X_{i,j} \times X_{k,\ell} \to X_{i+k,j+\ell}, \tag{6}$$

$$\nabla : X_{i+k,j+k} \to X_{i,j}. \tag{7}$$

There is a transitivity axiom for (7), commutativity, associativity and unitality axioms for (5), (6), and a distributivity axiom relating (6), (7) (see [2]).

Let us now discuss 'pseudo'. In the example we want to consider, I would be the set of all finite sets of copies of S^1 (equivalently, just finite sets), and $X_{s,t}$ the set of all worldsheets with a bijection from the set of inbound (resp. outbound) boundary components to s (resp. t). We see clearly that this cannot be a CMC under disjoint union and gluing, since, as well known, for example finite sets cannot form a commutative monoid. We note however that I forms a groupoid, and X can be considered as a strict functor from $I \times I$ to groupoids. Moreover, the operations $+$ on I can be defined as functorial, and (5), (6), (7) can be defined as functorial and strictly natural in I. For each axiom (relation among the operation), we moreover get a *coherence isomorphism* in the groupoid involved. These coherence isomorphisms are subject to *coherence diagrams*. The correct rules for creating coherence diagrams are somewhat tricky (as an incorrect choice might either miss some diagrams, or force the structure to be strict). For I, the correct choice of diagrams are the ones for a symmetrical monoidal category: philosophically, words in a finite set of variables can be formed using the operations. Every variable must be used exactly once. Then one word w_1 may be converted to another word w_2 by repeatedly using the relations (with substitutions allowed). When w_1 can be converted to w_2 in two different ways, a coherence diagram arises.

On the level of X, we may similarly form words out of the operations (5), (6), (7) and variables to be chosen from the X's. Note however that we also have another set of variables involved, to be chosen from I, and the subscripts of the X's from which the X-variables are taken must be pairs of words in the I-variables made using the operations in I, in order for our abstract word in the X-variables and operations to be defined. Our rule is that each of the X-variables (but not the I-variables) must be used exactly once in the word. Once this is established, the situation is similar as for I: we may process one word in the X-variables to another using the relations (axioms), and when this can be done in more than one way, we obtain a coherence diagram.

Pseudo-structure of this kind have pseudo limits (as shown in [1]), so they can be used to make stacks. A stack on a site with values in a 2-category with pseudo limits is defined by assigning to each object of the site functorially an object of *sections* with values in the 2-category with the requirement that Grothendieck covers go to pseudo limits (the usual cocycle condition arises, since the pseudo-limits are only defined up to equivalence). The site considered in this note will usually be finite-dimensional complex manifolds and open covers.

We consider the SPCMC of worldsheets $\mathcal{C}$ over the stack of (finite-dimensional) complex manifolds. For a Riemann surface X, we have an SPCMC $\mathcal{C}_X$. The underlying stack of commutative monoids has sections over M equal to covering spaces $\tilde{M}$ of M together with real-analytic maps

$$\tilde{M} \times S^1 \to X \tag{8}$$

which are injective on fibers. Morphisms are deck transformations which preserve the map (8). The objects of the SPCMC itself are bundles

$$\xi : E \to M \tag{9}$$

together with an additional map

$$\phi : E \to X \tag{10}$$

where ϕ is injective on fibers and the fibers of (9) are Riemann surfaces with given bundles of inbound and outbound boundary components similarly as in [4]. There is an obvious forgetful map $\mathcal{C}_X \to \mathcal{C}$.

For modular functors, we will consider the SPCMC $C(\mathbb{C}_2)$ where $\mathbb{C}_2$ denotes the symmetric bimonoidal category of finite-dimensional vector spaces over $\mathbb{C}$ and isomorphisms, and the operations of direct sum and tensor product. This is defined in [5] (where one defines, more generally, $C(\mathcal{M})$ for any finite dimensional free pseudo module over $\mathbb{C}_2$) . In the special case we are considering here, the underlying pseudo commutative monoid of $C(\mathbb{C}_2)$ is the category of finite sets. The category of sections over a point over a pair of sets S, T in $\mathbb{C}_2$ and the operations are the identity.

It will be to our advantage to consider this as a *topological stack*, which means that the morphisms in a category of sections over a given M are topologized by the topology on morphisms of vector spaces.

For CFT's, it will be to our advantage to vary the Hilbert space. This means that we will consider the SPCMC $C'(\mathbb{C}_2)$ whose sections over M over sets s, t are given by a holomorphic (finite-dimensional) vector bundles V on M, a Hilbert space H, and a trace-class map

$$V \to \hat{\bigotimes_{s}} H^* \hat{\otimes} \hat{\bigotimes_{t}} H. \tag{11}$$

Morphisms of CFT's which we will consider will fix modular functor, so morphisms from (V, H) to (V, K) consist of isomorphisms $H \cong K$ which intertwine the map (11) in the obvious way. These morphisms are topologized in the obvious way (using the norm topology on Hilbert space isomorphisms).

In this paper, we will only consider modular functors with one label (over $\mathbb{C}_2$), so the words "with one label" will be omitted from our terminology. By a modular functor (resp. CFT) on $\mathcal{C}_X$ we mean a morphism of SPCMC's $\mathcal{C}_X \to C(\mathbb{C}_2)$ (resp. $\mathcal{C}_X \to C'(\mathbb{C}_2)$) which on the underlying commutative monoid are given by forgetting the map (8). The corresponding notions for $\mathcal{C}$ are defined by considering $X = *$. For modular functors M on $\mathcal{C}$, we add the requirement that for the unit disks D^+,D^- $M_{D^\pm}$ be a complex line with a distinguished non-zero element $\epsilon_\pm$, and that morphisms preserve this element. For CFT, we add the requirement that the map (11) (which we denote by U) always be injective.

Lemma 1. *For a $\mathbb{C}_2$-modular functor M over $\mathcal{C}_X$, and for any worldsheet Σ, M_Σ is a complex line. Further, if $X = *$, M has no automorphisms except the identity.*

Proof: For 1-dimensionality, simply note that by the gluing axiom for modular functors, and our assumptions about $D^\pm$, M_Σ is 1-dimensional for any pair of pants, and hence again for every Σ by gluing. For automorphisms, by our assumption, an automorphism is Id on $M_{\mathbb{P}^1}$, and hence again on any disk, and hence any pair of pants, and hence any worldsheet. □

Remark: If we relax our assumptions on $D^{\pm}$, there will be non-trivial automorphisms of modular functors. For example, if we only assume that the automorphism is Id on M_{D^+}, then for any non-zero complex number λ, there is an automorphism which on Σ is $\lambda^{\chi-\partial_+ +\partial_-}$ where Σ has Euler characteristic χ, and ∂_+ (resp. ∂_-) outbound (resp. inbound) boundary components.

Now we make the categories of modular functors and CFT's into topological categories. (The category of $\mathcal{C}$-modular functors is discrete by Lemma 1.) Note that morphisms are specified by giving an element of a certain topological space on every object of $\mathcal{C}_X$. More concretely, for sections over an X we have the space of isomorphisms of the appropriate vector bundles and an isomorphism of the appropriate Hilbert spaces. Call this space G_X. We can also consider G_Y for a subspace Y of X. Now let the basis of the set of open sets of morphisms be the inverse images of G_Y for all compact subsets of all finite-dimensional manifolds X.

The topological category $\mathcal{A}$ of CFT's $\mathcal{C}$ then maps into the category $\mathcal{B}$ of CFT's $\mathcal{C}_{\mathbb{C}}$ over modular functors pulled back from $\mathcal{C}$ (as noted, morphisms take leave the modular functor fixed). The map is given by taking the constant CFT. Now both $\mathcal{A}$ and $\mathcal{B}$ are fibered over the discrete category $\mathcal{MF}$ of modular functors over $\mathcal{C}$. Each of these fibered categories can be taken as input into the machine of Gomez [3]. The fiber of the resulting E_∞ ring spectra is one version of our construction.

It is also possible to consider a modification of these concepts which is closer to the language of [4] and makes it easier to construct examples. At this point, we do not have a proof that both versions of the definition are equivalent. The modification is to define a *special* modular functor (resp. CFT) as the following set of data: A modular functor (resp. CFT) on $\mathcal{C}$, a finite set $S \subset \mathbb{C}$ (called the set of *punctures*), and a modular functor (resp. CFT) on the sub-SPCMC of $\mathcal{C}_{\mathbb{C}}$ where on the commutative monoid we restrict to objects whose image under (8) is disjoint with S, and on the 2-level we restrict to objects whose image under (10) has boundary disjoint with S. We further require that for objects on the sub-SPCMC where on the 2-level we restrict to worldsheets such that the image of (10) is disjoint with S, the data is given by pulling back the modular functor (resp. CFT) structure on $\mathcal{C}$. We also identify such special modular functor or CFT with one obtained by adding any additional finite number of punctures, and consider the resulting special modular functor resp. CFT equal. Topology for the category of special modular functors resp. CFT's is defined in the same way as for general modular functors resp. CFT's over $\mathcal{C}_{\mathbb{C}}$.

It is also helpful to further enlarge the category of CFT's by enlarging the set of morphisms. We will call this *special CFT's on* $\mathcal{C}_E$ *with weak convergence*. In this case, we require for a morphism only that to each analytic Jordan curve disjoint from the punctures, we are given a weak isomorphism of Hilbert spaces, which is an injective map

$$\psi : H \to \hat{K} \tag{12}$$

where the target is the completion of K with respect to the $\mathcal{C}$-CFT weights, and we assume that there are well defined trace class maps

$$U(A_r)\psi : H \to K \tag{13}$$

where A_r is the standard annulus $0 < |r|| < 1$, and (12) is the weight space-wise limit of (13) as $r \to 1$. It is easy to show that such morphisms have well defined composition.

Our objective is to plug in these definitions to the infinite loop space machine of [3]. The key point for doing that is the following

Lemma 2. *Let H_1, H_2 be two CFT's on the same SPCMC $\mathcal{C}$ or $\mathcal{C}_X$ with the same modular functor over $C(\mathbb{C}_2)$. Then there is a natural CFT structure on $H_1 \oplus H_2$.*

Proof: Given an object X of the source SPCMC over a pair of the underlying pseudocommutative monoid which forgets to a pair of sets s, t which represents a connected worldsheet, if the value of the corresponding modular functor is M_X, the two CFT's produce maps

$$U_i(X) : M_X \to \hat{\bigotimes_s} H_i^* \hat{\otimes} \hat{\bigotimes_t} H_i. \tag{14}$$

We want to produce a map

$$U(X) : M_X \to \hat{\bigotimes_s} (H_1 \oplus H_2)^* \hat{\otimes} \hat{\bigotimes_t} (H_1 \oplus H_2). \tag{15}$$

To this end, we note that since the dual $(?)^*$ commutes with $\oplus$, there are natural maps ϕ_i from the right hand side of (14) to the right hand side of (15). We take the sum of these maps.

A general object X is a direct sum of finitely many objects X_j, $j \in J$, each representing a connected worldsheet. We let

$$U(X) = \bigotimes_j U(X_j). \tag{16}$$

Since this definition obviously satisfies $U(X \amalg Y) = U(X) \otimes U(Y)$, we need to prove that it preserves gluing. This means that when $i \in s$ and $i' \in t$, and $\check{X}$ is obtained from X by gluing i to i', we need to show that

$$U(\check{X}) = tr(U(X)). \tag{17}$$

To this end, note that by definition, $U(X)$ is the sum of products

$$\otimes U_{k(j)}(X_j) \tag{18}$$

for some function $k : J \to \{1, 2\}$. By the definition of trace, the trace of (18) can be nonzero only if

$$k(j) = k(j') \tag{19}$$

where i is a boundary component of X_j and i' is a boundary component of $X_{j'}$. If $j = j'$, (19) is automatic, the number of connected components does not change, and by definition (17) then follows simply from the additivity of trace. If $j \neq j'$, then the number of connected components of $\check{X}$ is one less than the number of connected components of X, but the contribution of (18) can be non-zero only when (19), so once again, (17) follows from the additivity of the trace. □

Note that there is a canonical embedding functor Φ from the category $\mathcal{A}$ of CFT's over $\mathbb{C}$ to the category $\mathcal{B}'$ whose objects are CFT's over $\mathcal{C}$ and morphisms are special morphisms of their pullbacks to $\mathcal{C}_{\mathbb{C}}$ with weak convergence (and with rigid modular functor). Both categories $\mathcal{A}$ and $\mathcal{B}'$ are then fibered over the discrete category $\mathcal{MF}$ of modular functors, and the machine of Gomez [3] then gives rise to E_∞ ring spectra $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ together with a map ϕ of E_∞ ring spectra induced by the functor Φ.

Definition 3. *Let $\mathfrak{E}$ be the E_∞-ring spectrum obtained as a fiber of the map*

$$\phi : \mathcal{E}_{\mathcal{A}} \to \mathcal{E}_{\mathcal{B}}.$$

3. The basic properties

We begin with (4) (or more precisely its certain weakening which we can construct). To this end, it is helpful to extend the notion of stringy isomorphism and morphism of modular functor by allowing "variation of label". A special modular functor on $\mathcal{C}_{\mathbb{C}}$ associated with a given modular functor over $\mathcal{C}$ is equivalent data to a finite set of punctures S and a complex line L_x for each $x \in S$. (When $L_x = \mathbb{C}$, we may delete x from S.) Here we need to be more precise about what we mean by a "complex line". Specifically, we need the lines to form a topological space which is homotopy equivalent to $\mathbb{C}P^\infty$. Further, the space must form a strict monoid with respect to an operation $\cdot$ which is to be isomorphic to the tensor product, subject to the usual coherence diagrams (one for associativity, and one for left and right unitarity each). This can be achieved as follows: we can take as the "space of lines" the projective unitary group $PU(H)$ where H is our Hilbert space. This group possesses a central extension by $\mathbb{C}^\times$, which we can view as a line bundle on $PU(H)$. Let the line over g be L_g. Then define

$$L_g \cdot L_h := L_{gh}.$$

Actually, $PU(H)$ is not a complex manifold, so it is more convenient for us to replace it by its "complexification", the space $P'GL(H)$ of weakly convergent dense injective maps (in the above sense, i.e. after composing the result with U_{A_r} for an annulus A_r), which also have a dense injective inverse. The topology is above: for an analytic Jordan curve c bounding a holomorphic image of D and not containing an element of S, we let L_c be the tensor product of $(L_x)^i$ for all $x \in S$ where i is the index of

c with respect to x. The topology is by convergence on compact subsets of analytic Jordan curves.

We now see clearly that the space $\mathfrak{S}$ of special modular functors on $\mathcal{C}_{\mathbb{C}}$ is homotopy equivalent to $K(\mathbb{Z}, 4)$. Further, we have a group of *modular isomorphisms* acting on $\mathfrak{S}$ which consists of the same amount of data, namely a finite set of punctures S and a complex line L_x for each $x \in S$. The topological abelian group acts by tensor product of lines. Because of this, we will denote the topological abelian group also by $\mathfrak{S}$, and interpret its action on special modular functors as $\mathfrak{S}$ acting on itself by translation.

We can now consider the space $\mathfrak{C}$ of special CFT's over $\mathcal{C}_{\mathbb{C}}$ with modular functors allowing variation of labels. The objects consist of an element of $\mathfrak{S}$, a Hilbert space H, a CFT (M, U) over $\mathcal{C}$, and for a worldsheet Σ embedded into $\mathbb{C}$ with boundary disjoint from the punctures, a trace map with trace class image

$$M_\Sigma \otimes \bigotimes_c L_c \to \bigotimes_c H^{(*)}. \tag{20}$$

Here c are the boundary components of Σ, and $H^{(*)}$ means H^* or H depending on whether c is inbound or outbound. We further require that (20) be equal to U_Σ if Σ contains no punctures.

One essential point is that there is a canonical group of *stringy isomorphisms* acting on the space $\mathfrak{C}$ of special CFT's with variation of labels associated with a CFT H over $\mathcal{C}$. A stringy isomorphism consists of an element of $\mathfrak{S}$ (a modular functor with variation of labels), and maps

$$\phi_c : L_c \otimes H \to H \tag{21}$$

such that (21) intertwines with U_Σ for any worldsheet Σ embedded in $\mathbb{C}$ disjoint with S. Denote by $\mathfrak{G}$ the monoid of weakly convergent analogues of stringy isomorphisms (in the obvious sense).

It is worth noting that, as observed in [4], in certain cases of interest, the obvious map

$$Aut(H) \to \mathfrak{G} \tag{22}$$

factors through a contractible group Γ. For example, when H is the level 1 CFT associated with the group E_8, then one can consider the contractible group of meromorphic maps $\mathbb{C} \to (E_8)_{\mathbb{C}}$. The required structure is provided by the projective action of LE_8 on H. The lines are supplied by the universal central extension of LE_8. In more detail, the projective representation

$$\psi : (LE_8)_{\mathbb{C}} \to P'U(H) \tag{23}$$

by action of currents on the conformal field theory defines the universal central extension, so in our setup, we can assign to $g \in LE_8$ the line $L_{\psi(g)}$ (see above). For a meromophic map $f : \mathbb{C} \to (E_8)_{\mathbb{C}}$, consider a circle c (homothetic to the identity parametrization of S^1) with center in one of the poles or zeroes x of f, such that all the other poles or zeroes are in the exterior of c. Then let we take the line $L_{\psi(g)}$ where g is the restriction of f to c.

So we have $\mathfrak{G}$ acting on $\mathfrak{C}$ mapping to $\mathfrak{S}$ acting on itself. The fiber of this construction is a group $\tilde{\mathfrak{G}}$ acting on $\tilde{\mathfrak{C}}$, but it is easily seen to be isomorphic to the groupoid of special CFT's based on H, as considered in the last section. Now we have an obvious canonical map

$$Aut(H) \to \tilde{\mathfrak{G}}. \tag{24}$$

Note that in the E_8-example of the last paragraph, by our observations, the restriction of the homomorphism (24) to E_8 factors through the fiber

$$\Gamma \to \mathfrak{S} \tag{25}$$

which is $K(Z, 3)$. This is the sense in which (24) can be considered, at least conjecturally, as a version of (4).

It should be pointed out that the image of the map (24) actually lies in the isotropic group $\tilde{\mathfrak{G}}_0$ of the pullback of the CFT H with respect to $\tilde{\mathfrak{G}}$. Nevertheless, we claim that the inclusion

$$\tilde{\mathfrak{G}}_0 \subset \tilde{\mathfrak{G}},$$

since the space of CFT's on $\mathcal{C}_{\mathbb{C}}$ without variation of modular functor with its natural topology is actually contractible: such theory is specified by a finite set of punctures, and for a holomorphic image of the unit disk under a map sending 0 to each puncture x, an element u of H which, as a sum of weight components, decays faster than exponentially (so one has for $U(A_r)^{-1}u \in H$ for every standard annulus A_r, $0 < ||r||, 1)$. In any case, the space of such elements is contractible.

Getting a map (1) in our definition is almost trivial. Even if we disregard the fibration, i.e. fiber our category of $\mathcal{C}$-CFT's only over natural numbers equal to the central charge, for a conformal field theory H we have a map

$$Aut(H) \to \prod GL(V_n) \tag{26}$$

where V_n is the n-weight subspace of H. This defines a functor from our category to the category fibered over the discrete category of natural numbers, where over each number the fiber is the category of sequences (indexed by natural numbers) of finite-dimensional complex vector spaces. This is a morphism of the fibered symmetric bimonoidal categories of [3]. Addition and multiplication is given by the direct sum and graded tensor product. Passing to E_∞ ring spectra gives (1).

The map (3) now follows from the example to our construction of (a weaker version of) (4). To get to the 0 fiber, just divide by the constant map. It is worth commenting that the rational cohomology of BE_8 is

$$\mathbb{Q}[\alpha_1, ..., \alpha_8]$$

where the dimensions of the α_i's are twice the degrees $d_1, ..., d_8$, which are

$$2, 8, 12, 14, 18, 20, 24, 30.$$

By rational homotopy theory, some non-zero integral multiple of each α_i is realized by an element of $\pi_{2d_i}(BE_8)$. Clearly, we can lift to $\widetilde{BE_8}$ when $i > 1$. We showed in [4] that under the map (1), α_2 goes to a nonzero element of

$$\Delta/g_2, \tag{27}$$

and α_3 goes to a non-zero element of

$$\Delta. \tag{28}$$

We see that (27) and (28) are modular of the right weight, but (27) has a singularity. Remarkably, the image of α_1 is not modular.

It is worth commenting that the way (27) and (28) are calculated in [4] leads to the following conjecture, which, as far as we know, is not proved in the literature: Let L be an even unimodular lattice of dimension n and let

$$\theta_L(\tau, u_1, ..., u_n) \tag{29}$$

be its θ-series. Putting $q = e^{2\phi i\tau}$, (29) can be interpreted as a map

$$(\mathbb{Q}[u_1, ..., u_n]^{Aut(L)})^\vee \to \mathbb{Q}[[q]]. \tag{30}$$

(By $?^\vee$ we mean the dual.) Now compose (31) with the inclusion of the dual of the submodule of indecomposable elements, to get a map

$$(Q\mathbb{Q}[u_1, ..., u_n]^{Aut(L)})^\vee \to \mathbb{Q}[[q]]. \tag{31}$$

The left hand side is further naturally graded by assigning each u_i degree 1. Then the conjecture states that the image of a homogeneous element of degree $\ell > 2$ under (31) is a modular form of weight

$$\ell + \frac{n}{2}. \tag{32}$$

Remarkably, this fails for $\ell = 2$! From the point of view of CFT, the second summand of (32) corresponds to shift by central charge of the lattice CFT involved. However, at the moment we do not know that these forms are actually realized by homotopy classes of our spectrum $\mathfrak{E}$.

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