ON KONTSEVICH'S HOCHSCHILD COHOMOLOGY CONJECTURE

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1. INTRODUCTION

A conjecture of Deligne stated that the Hochschild cohomology complex of an associative algebra has a natural structure of a 2-algebra, i.e. an algebra over the chain complex version of the 2-cube operad. This indicated a remarkable connection between the deformation theory of associative algebras, and the geometry of configuration spaces of points in the plane. There are several known proofs of Deligne's conjecture, see [19], [20], [21], [15], [10]. The purpose of the present paper is to prove a generalization conjectured by Kontsevich [9], calling for an analogue of Deligne's conjecture for algebras over the little k-cube operad.

The first problem is to define a suitable generalization of the Hochschild cohomology complex. Kontsevich [9] proposes to do this by modifying the Quillen cohomology complex, but that approach forces some restrictions (in fact, it only seems to work for the little k-cube operad, and, as stated, only in chain complexes over fields of characteristic 0). A key feature of our approach is that we give a completely natural definition of the Hochschild cohomology complex, not restricted to those situations. In fact, all of our constructions in principle work for any operad, and in a closed symmetric monoidal category ([13]). Nevertheless, to avoid technical problems, we shall still stick to specific cases. Namely, in the statement of our theorems, we shall assume that $\mathcal{B}$ *is the category of sets, or K-modules where K is a field* (not necessarily of characteristic 0). There is at least one substantially different case of interest, namely the case of spectra (S-modules [5]). However, homotopical algebra in that case is more difficult, and will not be discussed here.

Now consider operads $\mathcal{C}$ in the category of simplicial sets. An example of special interest to us is the operad $\mathcal{C}_k$ which is the *set of singular simplices of the operad of little k-cubes* [14]. In the beginning of the next Section, we will introduce the notion of $\mathcal{C}$-algebras R in the category $s\mathcal{B}$, and $(\mathcal{C}, R)$-modules. Furthermore, for $(\mathcal{C}, R)$-modules M, N, we will construct a 'derived mapping object'

$$RHom_{(\mathcal{C},R)}(M, N)$$

in the homotopy category of $s\mathcal{B}$. Then our main result can be stated as follows:

Theorem 1. *Let R be a cofibrant $\mathcal{C}_k$-algebra (in $s\mathcal{B}$ as above). Then there is a functorial model of*

$$RHom_{(\mathcal{C}_k,R)}(R, R)$$

which is a $\mathcal{C}_{k+1}$-algebra.

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Tamarkin [18] previously proved a purely algebraic version of Theorem 1 for the Quillen cohomology complex [16] (also known as the deformation complex), working in the category of chain complexes rather than simplicial modules. His proof works over fields of characteristic 0, and does not include the statement of Deligne's original conjecture (the case of $k = 1$). It uses the fact that the little cube chain operad in that case is formal and Koszul.

Theorem 1 is based on the geometry of the little k-cube operad. Its proof will be completely 'derived' and based on a general principle (cf. [3]) that "commuting $\mathcal{C}_k$- and $\mathcal{C}_1$-structures give a $\mathcal{C}_{k+1}$-structure". In the case of Hochschild cohomology of R, the $\mathcal{C}_k$-structure is induced by the $\mathcal{C}_k$-structure of R, the $\mathcal{C}_1$-structure from the Yoneda product. Our proof applies to modules over fields of any characteristic (as well as sets), and does include the case $k = 1$. In the statement of the Theorem, the assumption that R be cofibrant should not be regarded as a restriction, since for non-cofibrant algebras the correct notion of Hochschild cohomology is obtained by first taking a cofibrant replacement: this leads to the right notion of Hochschild cohomology in the case $k = 1$.

The present paper is organized as follows: In the next Section, we shall reduce Theorem 1 to a much more general context, which may be of independent interest as a generalization of the Kontsevich conjecture. We will introduce the notion of operads fibered over a given operad $\mathcal{C}$ in simplicial sets. We will also introduce a certain □-product of operads, and in Theorem 3, we will state that for a certain class of operads Q fibered over $\mathcal{C}$, which we call *special*, there is a notion of Hochschild cohomology object constructed from Q, and that moreover this object has the structure of a $\mathcal{C}□\mathcal{C}_1$-algebra. It should be noted that our □-product of operads is based on ideas analogous to those of G. Dunn [3].

There are three facts which together will reduce the proof of Theorem 1 to Theorem 3: First, a $\mathcal{C}_k$-algebra gives rise to an example of a special operad fibered over $\mathcal{C}_k$. Second, the two ensuing notions of Hochschild cohomology coincide. Third, we have $\mathcal{C}_k□\mathcal{C}_1 \simeq \mathcal{C}_{k+1}$. All these statements are proven in Section 2 with the exception of the 'special' property: that is technical, and left to Section 6.

In Sections 3, 4, we will describe the technical machinery used to prove Theorem 3. This technique can be separated into two steps: First, in Section 3, we shall consider "lax algebras" over an operad $\mathcal{C}$ and show how they may be turned into strict algebras over a different, but weakly equivalent, operad $\overline{\mathcal{C}}$. In Section 4, we shall introduce an additional, "vertical" category structure on a lax algebra, which will allow us to get a $\overline{\mathcal{C}}□\mathcal{C}_1$-algebra. Both constructions must be discussed also in categories "enriched" over a given category $\mathcal{B}$. In Section 5, we will apply the techniques of Sections 3, 4 to our main example, which will give the proof of Theorem 3.

In Section 7, we describe some connections of the present paper with broader considerations. In particular, we discuss Quillen cohomology, Koszul duality and how our approach relates to the ideas of Kontsevich [9] on linking Hochschild and Quillen cohomology.

2. Special operads fibered over $\mathcal{C}$

In this Section, we shall reduce Theorem 1 to another statement, which can be phrased in a more general context. However, we begin by filling in the missing definitions in the statement of Theorem 1.

Recall [13] that a *closed symmetric monoidal category* is a category $\mathcal{B}$ with a symmetric monoidal structure $\boxtimes$ and an 'internal Hom' functor $Hom : \mathcal{B}^{Op} \times \mathcal{B} \to \mathcal{B}$ with a natural bijection

$$(1) \qquad Mor(A \boxtimes B, C) \cong Mor(A, Hom(B, C)),$$

satisfying certain axioms. We shall work in the category $s\mathcal{B}$ of simplicial objects in $\mathcal{B}$.

For a set S and an object (or morphism) X of $\mathcal{B}$, let

$$S \otimes X = \coprod_S X.$$

We say that an object R of $\mathcal{B}$ is a $\mathcal{C}$-algebra for an operad $\mathcal{C}$ in sets if there are structure maps

$$(2) \qquad \mathcal{C}(n) \otimes (R \boxtimes ... \boxtimes R) \to R$$

satisfying the usual axioms (see e.g. [11]). Now an important point is that here (and analogously in other places below), the same definition may be used for an operad in simplicial sets and an object of $s\mathcal{B}$ (this simply means that maps (2) exist on each simplicial level, and are natural with respect to simplicial structure). For a $\mathcal{C}$-algebra R, a $(\mathcal{C}, R)$-module is an object M of $\mathcal{B}$ together with structure maps

$$(3) \qquad \mathcal{C}(n) \otimes (M \boxtimes R \boxtimes ... \boxtimes R) \to M$$

satisfying the usual axioms (c.f. [11]). If C_R is the monad in $s\mathcal{B}$ defining free $(\mathcal{C}, R)$-modules, then define for $(\mathcal{C}, R)$-modules M, N,

$$Hom_{(\mathcal{C},R)}(M, N)$$

as the equalizer of the two obvious maps

$$Hom(M, N) \; \overset{\to}{\underset{\to}{}} \; Hom(C_R M, N).$$

(One is induced by $C_R M \to M$, the other by the natural map $Hom(M, N) \to Hom(C_R M, C_R N)$, composed with the map induced by the map $C_R N \to N$.)

We will be considering Quillen (closed) model structures on certain categories, which will be needed to do homotopy theory in those categories. A (closed) model structure on a complete cocomplete category consists of three classes of morphisms called fibrations, cofibrations and equivalences. (The word closed may be omitted as it carries no meaning.) All the information on closed model structures needed in this paper, and all the methods of construting closed model structures we shall need can be found in [17], [4]. One should remark that on many categories there are many closed model structures which have the same equivalences and hence lead to the same homotopy theory. However, when doing constructions, one typically needs to fix a closed model structure in order to control the homotopy behavior of the objects produced by the construction.

The most basic case needed is the category of simplicial sets (see [4], 11.1). In this category, there is a closed model structure where cofibrations are injective maps (more precisely sequences of injective maps), and equivalences are maps of simplicial sets which produce homotopy equivalences after simplicial realization. (It then follows that fibrations are so called Kan fibrations, but that is less important to us.) Next, recall ([4], 11.2) that a Quillen model structure in $s\mathcal{B}$ is obtained as follows: Let U be the forgetful functor from $s\mathcal{B}$ to simplicial sets. Then a morphism f in $s\mathcal{B}$ is a fibration or equivalence if and only if Uf is a fibration (equivalence). We say that fibrations and equivalences are *created* by the forgetful functor U.

Remark: Actually, it turns out that in the two cases we consider in this paper, f is a cofibration if and only if Uf is a cofibration also, i.e. cofibrations are also created by U: the case of simplicial sets is tautological, and in the case of simplicial vector spaces that category is an abelian category, equivalent to the category of ≥ 0-graded chain complexes by functors which preserve injections and surjections, and hence every injection or surjection which induces an equivalence splits. Additionally, all simplicial surjections are fibrations, but cofibrations are characterized as maps having the left lifting property with respect to fibration equivalences, so cofibrations are simplicial injections also.

There is also a canonical Quillen model structure on the category of $\mathcal{C}$-algebras (with equivalences and fibrations same as in $s\mathcal{B}$, i.e. created by the forgetful functor), and for a cofibrant $\mathcal{C}$-algebra R, there is a canonical Quillen model structure on the category of $(\mathcal{C}, R)$-modules (again with equivalences and fibrations same as in $s\mathcal{B}$). This is proven by a "small object argument", and the proof works for categories of algebraic structures in $s\mathcal{B}$ of very general kinds. (The small object argument is described in [4], 7.12). We shall refer to representatives of an $s\mathcal{B}$-equivalence class as models.

With the Quillen model structure established, now recall that an object X is called cofibrant (resp. fibrant) if the map from the initial object to X (resp. from X to the terminal object) is a cofibration (resp. fibration). A *cofibrant replacement* (resp. fibrant replacement) of an object M (resp. N) is a map

$$M' \to M$$

(resp. $N \to N'$) which is a fibration equivalence (resp. cofibration equivalence) and M' (resp. N') is cofibrant (resp. fibrant). We define

$$(4) \qquad\qquad RHom_{(\mathcal{C},R)}(M, N)$$

as $Hom_{(\mathcal{C},R)}(M', N')$ where M' is a cofibrant replacement of M and N' is a fibrant replacement of N. In our cases, cofibrant and fibrant replacement can be made functorial, so (4) is well defined. Nevertheless, cofibrant and fibrant replacements are not *canonical*, and hence it is appropriate to address the question of comparing the different $RHom$'s when different selections are made.

To this end, one uses the following technique. Let I be the standard simplicial model of the unit interval. Then we have objects of the form

$$I \otimes M$$

which are cylindrical objects (see [4], 4.1) in the sense that the two maps

$$i_0, i_1 : M \to I \otimes M$$

induced by the inclusions of the endpoints to I are equivalences and $i_0 \coprod i_1 : M \coprod M \to I \otimes M$ is a cofibration. We define a *homotopy* of two maps $f_0, f_1 : M \to N$ to be a map $I \otimes M \to N$ which, when composed with i_j, gives f_j. This is a particular example of what is known as a Quillen left homotopy, but the present notion has more features which we will find useful. In particular, the functor $I \otimes ?$ has a right adjoint which we will denote by $F(I, ?)$, and also the internal *Hom*-functor $Hom_{(\mathcal{C}, R)}$ obeys the relation

$$(5) \qquad Hom_{(\mathcal{C}, R)}(I \otimes M, N) \cong Hom_{(\mathcal{C}, R)}(M, F(I, N)) \cong F(I, Hom_{(\mathcal{C}, R)}(M, N)).$$

To be precise, the $F(I, ?)$ on the right hand side of (5) is in $s\mathcal{B}$ rather than the category of $(\mathcal{C}, R)$-modules. Note, however, that since $F(I, ?)$ is a limit, the forgetful functor from $(\mathcal{C}, R)$ to $s\mathcal{B}$ preserves $F(I, ?)$, so it is given by the same construction in $s\mathcal{B}$ as in $(\mathcal{C}, R)$-modules. All this is formal. Additionally, it is true in our case that $F(I, N)$ is a co-cylindrical object (satisfying properties dual to cylindrical object; also known as path object, see [4], 4.12) if N is fibrant (since, again, this is true in $s\mathcal{B}$). It follows that for any other cofibrant replacement $M'' \to M$ there is a comparison map

$$(6) \qquad\qquad\qquad M' \to M''$$

commuting with the specified maps into M, and moreover unique up to homotopy (in our sense). Hence, by the same principle, we also abtain a map

$$M'' \to M',$$

and the compositions are homotopic to the identity. We call this a homotopy equivalence of $(\mathcal{C}, R)$-modules. But then applying $Hom_{(\mathcal{C}, R)}(?, N')$, we obtain a homotopy equivalence in $s\mathcal{B}$, which is an equivalence. The treatment of fibrant replacements is adjoint.

We shall now turn to the reduction of Theorem 1 to another statement.

Definition: Let S be a simplicial set, i.e. a functor $\Delta^{Op} \to Sets$. Then S can also be viewed as a category $\overline{S}$ with objects $\coprod_n S_n$, and morphisms $\phi : s \to t$ where $\phi \in Mor(\Delta^{Op})$, $s \in S_n$ for some n, and $\phi(s) = t$. Let Cat be any category. Then a *sCat-object fibered over S* is, by definition, a functor

$$F : \overline{S} \to Cat.$$

We shall write $F_s = F(s)$ for $s \in S_n$. For a map $i : S \to T$ of simplicial sets, we have a functor i^* from $sCat$-objects over T to $sCat$-objects over S, given by

$$i^*(F) = F \circ \overline{i}.$$

We shall also make use of the left adjoint to i^*, which we shall denote by $i_\sharp$.

Note that $sCat$-objects fibered over the constant simplicial set $*$ are precisely $sCat$-objects, which helps justify the terminology.

Specifically, we will now be interested in the case $Cat = s\mathcal{B}$ as above. Clearly, for every pair of objects X, Y of $s\mathcal{B}$ fibered over simplicial sets S, T, there is a canonical object $X \boxtimes Y$ fibered over $S \times T$.

Remark: Simplicial realization $|\ | : ss\mathcal{B} \to s\mathcal{B}$ is defined as the diagonal functor [11]:

$$|X|_n = X_{nn}.$$

By definition, we have

$$|X \boxtimes Y| = |X| \boxtimes |Y|.$$

We will sometimes drop the symbol $|\ |$ from our notation.

Let $\mathcal{C}$ be an operad in simplicial sets. Then an *operad Q in $s\mathcal{B}$ fibered over* $\mathcal{C}$ consists of objects $Q(n)$ of $s\mathcal{B}$ fibered over $\mathcal{C}(n)$, with Σ_n-action, and unity for $n = 1$, and, for each of the compositions

$$\gamma : \mathcal{C}(k) \times \mathcal{C}(n_1) \times ... \times \mathcal{C}(n_k) \to \mathcal{C}(n_1 + ... + n_k),$$

a composition

$$\gamma_\sharp(Q(k) \boxtimes Q(n_1) \boxtimes ... \boxtimes Q(n_k)) \to Q(n_1 + ... + n_k)$$

satisfying the obvious axioms analogous to the operad axioms [14].

Now simplicial sets fibered over a simplicial set S are precisely simplicial sets X over S, i.e. arrows $X \to S$. A morphism in this category is a fibration, cofibration of equivalence if and only if it has the corresponding property in simplicial sets. If U_S is the forgetful functor from objects and morphisms of $s\mathcal{B}$ fibered over S to simplicial sets fibered over S, then we say that a morphism f in $s\mathcal{B}$ fibered over S is a fibration or equivalence if and only if $U_S f$ is a fibration or equivalence. By [4], again, this defines a Quillen model structure on the category of objects of $s\mathcal{B}$ fibered over S.

Finally, on operads (similarly as any type of algebraic structure) in objects and morphisms of $s\mathcal{B}$ fibered over $\mathcal{C}$ we consider the closed model structure taking as fibrations (resp. equivalences) sequences of maps $(A(n) \to \mathcal{C}(n))_n$ which are fibrations (equivalences) in the category of objects and morphisms of $s\mathcal{B}$ fibered over $\mathcal{C}(n)$.

Definition: Let, for any simplicial set S, $h = h_S : S \to *$ be the collapse map, and let

$$Z = B(\mathcal{C}(\ell), \mathcal{C}(1)^{\times \ell}, \mathcal{C}(0)^{\times \ell}).$$

We shall call an operad Q fibered over $\mathcal{C}$ *special* if, for every ℓ, the map

$$(7) \qquad\qquad B(Q(\ell), Q(1)^{\boxtimes \ell}, Q(0)^{\boxtimes \ell}) \to h_Z^* Q(0)$$

(where B is the two-sided bar construction), induced from the composition map

$$h_{Z\sharp} B(Q(\ell), Q(1)^{\boxtimes \ell}, Q(0)^{\boxtimes \ell}) \to Q(0)$$

is an equivalence.

Remark: If Q is fibrant in the category of $s\mathcal{B}$-operads fibered over $\mathcal{C}$ such that $\mathcal{C}(0) = *$, $\mathcal{C}(1) \simeq *$, and the unit inclusion

$$i : \{*\} \to \mathcal{C}(1)$$

is a cofibration equivalence (which we are assuming), then the counit map

(8) $$i^*Q(1) = h_\sharp i_\sharp i^*Q(1) \to h_\sharp Q(1)$$

is an equivalence. We note that (8) is a map of monoids in $s\mathcal{B}$. We shall denote

$$Q_1 = i^*Q(1).$$

Thus, for fibrant operads Q over $\mathcal{C}$, we can replace $Q(1)$ by Q_1 in (7). Of course, every operad in $s\mathcal{B}$ over $\mathcal{C}$ can be replaced by a fibrant model. We shall make use of this below.

Now for a monoid R in $s\mathcal{B}$, a *module* over R is an object M of $s\mathcal{B}$ with a map

$$R \boxtimes M \to M$$

satisfying the usual axioms. Clearly, R-modules are precisely algebras over a monad C_R of the form

$$C_R(X) = R \boxtimes X.$$

Thus, we have a canonical closed model structure on R-modules for any monoid R. As before, we define

$$Hom_R(M, N)$$

as the equalizer of the two obvious maps

$$Hom(M, N) \; \overset{\to}{\underset{\to}{}} \; Hom(C_R M, N),$$

and define $RHom_R(M, N) = Hom_R(M', N')$ where M' is a cofibrant replacement of M and N' is a fibrant replacement of N (with derived independence on the choice of M' and N', for $s\mathcal{B}$ as above). Note that an example of a Q_1-module in the preceding remark is $Q(0)$.

Now for two operads $\mathcal{C}, \mathcal{D}$ in simplicial sets, define an operad $\mathcal{C} \square \mathcal{D}$ as the quotient of the free operad $\mathcal{F}$ on $\mathcal{C} \coprod \mathcal{D} = (\mathcal{C}(n) \coprod \mathcal{D}(n))_n$ modulo identifying the $\mathcal{F}$-operad operations on objects of $\mathcal{C}$, $\mathcal{D}$ with the corresponding operations in $\mathcal{C}$, $\mathcal{D}$, (this includes units), and the following key relation: for $\alpha \in \mathcal{C}(m)$, $\beta \in \mathcal{D}(n)$,

(9) $$\alpha(\underbrace{\beta, ...\beta}_{m \text{ times}}) = \beta(\underbrace{\alpha, ...\alpha}_{n \text{ times}})\sigma$$

where σ is a certain permutation reordering terms. To describe this permutation, consider the "row by row" lexicographical bijection

$$\rho_1 : \{1, ..., m\} \times \{1, ..., n\} \to \{1, ..., mn\}$$

(i.e. $(11 \mapsto 1, 12 \mapsto 2, ..., 1n \mapsto n, 21 \mapsto n+1, ...)$), and the "column by column" lexicographical bijection

$$\rho_2 : \{1, ..., m\} \times \{1, ..., n\} \to \{1, ..., mn\}$$

(i.e. $(11 \mapsto 1, 21 \mapsto 2, ..., m1 \mapsto m, 12 \mapsto m+1, ...)$). The point is that on the left hand side of (9), the entries are ordered "row by row" whereas on the right hand side they are ordered "column by column". Since permutations on operads act on the right, we conclude that

$$\sigma = \rho_2 \circ (\rho_1)^{-1}.$$

Proposition 2. *There is a canonical map of operads*

$$(10) \qquad\qquad \phi : \mathcal{C}_k \square \mathcal{C}_\ell \to \mathcal{C}_{k+\ell},$$

which is a (term-wise) equivalence. An analogous result also holds if we work in the category of topological spaces (rather than simplicial sets).

We shall prove this at the end of this Section. Although this proposition is essentially equivalent to a theorem of Gerald Dunn ([3], Theorem 2.9), our technical setting is different, and we will find it easier to prove the result directly. We will now restate Theorem 1 as follows:

Theorem 3. *Let Q be a special fibrant operad in $s\mathcal{B}$ fibered over $\mathcal{C}$ where $s\mathcal{B}$ is as above and $\mathcal{C}$ is an operad in simplicial sets with $\mathcal{C}(0) = *$, $\mathcal{C}(1) \simeq *$. Then there is a model of*

$$(11) \qquad\qquad RHom_{Q_1}(Q(0), Q(0))$$

which has a natural structure of a $\mathcal{C}\square\mathcal{C}_1$-algebra.

A discussion is needed to see how Theorem 3 implies Theorem 1. Recall that in general for a monad C a C-functor D (i.e. a functor with a structure map $DC \to D$ satisfying the usual axioms), and a C-algebra X, we can define

$$D \otimes_C X$$

as the coequalizer of the two maps

$$DCX \; \overset{\to}{\underset{\to}{}} \; DX$$

given by the structure maps $DC \to D$, $CX \to X$. Now for an operad $\mathcal{C}$ in $s\mathcal{B}$, $\mathcal{C}$-algebras are algebras of the monad

$$CX = \coprod_{n \geq 0} \mathcal{C}(n) \otimes_{\Sigma_n} X^{\boxtimes n}.$$

We shall define C-functors D_ℓ as follows:

$$(12) \qquad\qquad D_\ell X = \coprod_{n \geq 0} \mathcal{C}(n + \ell) \otimes_{\Sigma_n} X^{\boxtimes n}.$$

The following statement is obvious upon a moment's reflection:

Proposition 4. *Let $\mathcal{C}$ be an operad in simplicial sets and assume in addition $\mathcal{C}(0) = *$ (the one point simplicial set). Let R be a $\mathcal{C}$-algebra. Then the object*

$$A(n) = D_n \otimes_C R$$

has the natural structure of an operad in $s\mathcal{B}$ fibered over $\mathcal{C}$.

Proof: Recall that because the functor $? \boxtimes X$ is $\mathcal{B}$ has a right adjoint, $\boxtimes$ is distributive under $\coprod$. Now we need to construct structure maps

$$\mathcal{C}(n + \ell) \otimes_{\Sigma_n} R^{\boxtimes n} \boxtimes$$

(13)
$$(\mathcal{C}(n_1 + m_1) \otimes_{\Sigma_{n_1}} R^{\boxtimes n_1}) \boxtimes ... \boxtimes (\mathcal{C}(n_\ell + m_\ell) \otimes_{\Sigma_{n_\ell}} R^{\boxtimes n_\ell})$$

$$\downarrow$$

$$(\mathcal{C}(n + n_1 + ... + n_\ell + m_1 + ... + m_\ell) \otimes_{\Sigma_{n_1 + ... + n_\ell}} R^{\boxtimes n + n_1 + ... + n_\ell})$$

compatible with $\otimes_C$. However, realizing that $\otimes$ is nothing but coproducts over various sets, distributivity applies to this case also, and we see that the map (13) is obtained just by applying operad composition in $\mathcal{C}$, and grouping the $\boxtimes$-powers of R. Compatibility with $\otimes_C$ is obvious from operad axioms. Additionally, the fibered structure is obtained by taking, for an element

$$x \in \mathcal{C}(n)_i$$

(i denotes the simplicial index) $A(n)_x$ to be the coproduct of all

$$\{y\} \otimes_{\Sigma_\ell} R^{\boxtimes \ell}$$

where $y \in \mathcal{C}(n + \ell)$ is such that

(14)
$$\gamma(y, 1, ..., 1, *, ..., *) = x$$

where γ is operad multiplication, and in (14), there are n 1's and ℓ *'s. $\qquad\square$

The reduction from Theorem 1 to Theorem 3 then follows from the following two results:

Lemma 5. *The category of $(\mathcal{C}, R)$-modules is equivalent to the category of $h_\sharp A(1)$-modules. This equivalence of categories carries R to $A(0)$. Moreover, if R is a cofibrant $\mathcal{C}$-algebra (with $s\mathcal{B}$ as above), this is a Quillen equivalence, i.e. passes on to an equivalence of Quillen homotopy categories.*

Proof: The equivalence of categories is established by the fact that both categories consist of algebras over the same monad:

(15)
$$X \mapsto h_\sharp A(1) \boxtimes X.$$

Indeed, it suffices to consider the case when R is a free $\mathcal{C}$-algebra, i.e. $R = CX$. Then

$$A(1) = D_1 \otimes_C R = D_1 X$$

$$= \coprod_{n \geq 0} \mathcal{C}(n + 1) \otimes_{\Sigma_n} X^{\boxtimes n}.$$

Now the free $(\mathcal{C}, R)$-module on M is

$$\coprod_{n \geq 0} \mathcal{C}(n + 1) \otimes_{\Sigma_n} (X^{\boxtimes n} \boxtimes M) = A(1) \boxtimes M.$$

(By abuse of notation, we treat $h_\sharp$ as the forgetful functor, so this is the same as (15).)

Now if R is a cofibrant $\mathcal{C}$-algebra, both Quillen model structures on the respective categories are defined in the same way.   □

Remark: Note that this is analogous to a method used by Zhu [22] for vertex operator algebras (see also [2]). Concretely, vertex operator algebras are close to the notion of algebra over a certain modification to the little 2-disk operad $\mathcal{D}$ (see [8]). One could elaborate a lot more on that, but in the rough analogy, the notion of module over a vertex operator algebra V corresponds to the notion of $(\mathcal{D}, V)$-module. Now Zhu [22] describes an associative algebra A with the property that, for a rational vertex operator algebra V, irreducible V-modules M are in bijective correspondence with irreducible A-modules. However, the algebra A is *not* a precise analogue of $h_\sharp A(1)$: in [22], the algebra A is finite-dimensional, and only acts on the top weight part of M.

Theorem 6. *Let $\mathcal{C} = \mathcal{C}_k$ be the little cube operad, and let R be a free $\mathcal{C}_k$-algebra. Let A be defined as above in Proposition 4. Then A is special (although not fibrant).*

We shall prove this theorem below in Section 6.

Remark: There are other examples of special operads. For example, working in spaces (one can get to simplicial sets by applying the singular set functor), let, for $e \in \mathcal{C}_k(n)$, and a based CW complex X,

$$(\Phi_k(X))_e = Map((e, \partial e), (X, *)).$$

Then there is a standard way to put a topology on

$$\Phi_k(X)(n) = \bigcup_{e \in \mathcal{C}_k(n)} \Phi_k(X)_e,$$

making $\Phi_k(X)$ an operad fibered over the topological version of $\mathcal{C}_k$. It can be shown that $\Phi_k(X)$ is special if X is $(k-1)$-connected.

On the other hand, it is easy to construct operads $\mathcal{C}$ in simplicial sets such that, for $R = CX$ the associated operad A fibered over $\mathcal{C}$ is not special: It suffices to take a free operad on a set (in the category of operads $\mathcal{C}$ with unit and $\mathcal{C}(0) = *$).

We shall conclude this section with a

Proof of Proposition 2: We shall first prove the statement for the category of topological spaces. In this case, let $\mathcal{C}_k$ denote the original topological space models of the little cube operads rather than the singular set model. In this setting, the map (10) is obtained by sending a configuration of n little cubes

$$(c_1, ..., c_n) \in \mathcal{C}_k(n)$$

to

$$(c_1 \times I^\ell, ...c_n \times I^\ell) \in \mathcal{C}_{k+\ell}(n),$$

and a configuration of n little cubes

$$(d_1, ..., d_n) \in \mathcal{C}_\ell(n)$$

to

$$(I^k \times d_1, ...I^k \times d_n).$$

We will not construct a homotopy inverse of (10) on the whole $\mathcal{C}_{k+\ell}(n)$, but instead on a certain subspace $\mathcal{C}_{k+\ell}(n)'$ which is weakly equivalent. To define this subspace, put $m = k + \ell$. We shall call an n-tuple of little cubes

$$(16) \qquad\qquad e = (e_1, ..., e_n)$$

which is an element of $\mathcal{C}_m(n)$ *small* is the following condition is satisfied: there exists a p-tuple of little cubes

$$f = (f_1, ..., f_p)$$

forming an element of $\mathcal{C}_k(p)$ and a q-tuple of little cubes

$$g = (g_1, ..., g_q)$$

forming an element of $\mathcal{C}_\ell(q)$ such that every little cube e_i lies in the interior of precisely one set

$$(17) \qquad\qquad f_h \times g_j,$$

and every set (17) contains at most one little cube e_i. The space $\mathcal{C}_m(n)'$ is the subspace of $\mathcal{C}_m(n)$ consisting of precisely all small n-tuples. Now we claim that the inclusion $\mathcal{C}_m(n)' \subset \mathcal{C}_m(n)$ is a homotopy equivalence. Indeed, let e be as in (16). Then, for $\lambda \in (0,1]$, define λe as the little cube configuration obtained by scaling each little cube e_i by a factor λ in its center. Then we know that $\lambda e \in \mathcal{C}_m(n)$, and if e is an element of $\mathcal{C}_m(n)'$, then so is λe. Furthermore, it is easy to see that for every e there exists a λ such that $\lambda e \in \mathcal{C}_m(n)'$ (the statement is true trivially if every little cube is replaced by one point, namely its center). By the Lebesgue number theorem, for every compact subset $K \subset \mathcal{C}_m(n)'$, there exists a $\lambda \in (0,1]$ such that $\lambda K \subset \mathcal{C}_m(n)'$. Furthermore, $t.Id$, $t \in [\lambda, 1]$ is a homotopy between K and λK, which moreover stays in $\mathcal{C}_m(n)'$ if $K \subset \mathcal{C}_m(n)'$. By the Whitehead theorem, the inclusion $\mathcal{C}_m(n)' \subset \mathcal{C}_m(n)$ is a weak equivalence, and hence in fact a homotopy equivalence, since $\mathcal{C}_m(n)$ is a CW complex.

Now we will construct a right inverse ψ to the map ϕ when restricted to $\mathcal{C}_m(n)'$. In fact, this construction is obvious: simply compose f with p copies of g, and each entry with either $*$ or the appropriate elements of $\mathcal{C}_k(1)$, $\mathcal{C}_\ell(1)$ to ensure that $\phi\psi = Id_{\mathcal{C}_m(n)'}$. It should be noted that the map is well defined and continuos, because its value on an element e does not depend on the choice of f, g: any two choices have a "common subdivision", which produces the same element by the fundamental relation (9). To be more precise, if A, B are two sets of disjoint little cubes in I^k, the *common subdivision* of A, B is

$$\{a \cap b | a \in A, b \in B\}.$$

(While ordering of the cubes of course matters in the operad structure, we do not have to specify it in this definition, as any two ordering are related by the symmetric group action, and hence any ordering will do.)

Note that we are not yet done: We must still produce a homotopy left inverse to ϕ. But now let

$$(\mathcal{C}_k \square \mathcal{C}_\ell)(n)' = \psi^{-1}(\mathcal{C}_m(n)').$$

First of all, note that the inclusion

$$(\mathcal{C}_k \square \mathcal{C}_\ell)(n)' \subset (\mathcal{C}_k \square \mathcal{C}_\ell)(n)$$

is a weak equivalence by the same argument as above: we may emulate the homotopy corresponding to multiplying e by t by composing an element of $(\mathcal{C}_k \square \mathcal{C}_\ell)(n)$ with n copies of $\gamma(t.1_k, t.1_\ell)$ where γ is operad composition, and $1_k \in \mathcal{C}_k(1)$, $1_\ell \in \mathcal{C}_\ell(1)$ are the unit elements. So we are done if we can show that $\phi\psi = Id$ on $(\mathcal{C}_k \square \mathcal{C}_\ell)(n)'$. But this is just a refinement of the above argument that ψ did not depend on the choice of f, g: one may form common subdivisions with the $\mathcal{C}_k$ and $\mathcal{C}_\ell$ elements u_i figuring in the definition of an element of $(\mathcal{C}_k \square \mathcal{C}_\ell)(n)'$, and use the relation (9) to show that the common subdivision produces the same element as using either f, g, or u_i. This concludes the proof of our statement for the category of topological spaces.

We shall now study what changes when we work in the category of simplicial sets, using the singular sets of $\mathcal{C}_k(n)$ etc. instead of the actual spaces. Much the idea is the same. For example, the construction of the map ϕ is got simply by applying the singular set functor to the space level ϕ (it is useful to note that $\square$ commutes both with the singular set functor, as well as realization of simplicial sets into spaces). However, when constructing the map ψ, we must adapt the definition of $\mathcal{C}_m(n)'$. In fact, we must introduce the notion of *small singular simplex* in $\mathcal{C}_m(n)$ as follows: if we represent the singular simplex by an n-tuple of singular simplices

$$(e_1, ..., e_n)$$

in the space of little cube, (i.e. for $t \in \Delta_N$, for some N, $e_i(t)$ is a little cube), then there must exist a *uniform* (i.e. independent of t) choice of f, g such that $e_i(t)$ satisfy the above condition in place of e_i for each t. This means, roughly, that the value of a small singular simplex at each t is required to be small, but also the values of the singular simplex must vary only by a "small" amount. We take $\mathcal{C}_m(n)'$ as the simplicial set of small singular simplices in $\mathcal{C}_m(n)$. Then the map ψ may be define completely analogously as in the case of spaces, by passing to singular sets.

It is, further, correct to think of ψ as a right homotopy inverse to ϕ, as it can be shown that $\mathcal{C}_m(n)' \subset \mathcal{C}_m(n)$ is an equivalence: this is a special case of a general theorem stating that for any open covering (U_i) of a space X, the inclusion of the sub-simplicial set of the singular set of X consisting of singular simplices whose images are in one of the U_i's is an equivalence. (In our case, X is the topological $\mathcal{C}_m(n)'$.)

However, we must still find a left homotopy inverse to ϕ. To this end, we will find it convenient to display the simplicial set functor S explicitly, to prevent confusion. Denote the j-fold barycentric subdivision of a simplicial set T by $T^{(j)}$. Then it is a standard fact (used for example in proving Eilenberg-Steenrod axioms for singular homology) that we have a canonical simplicial map

$$(18) \qquad\qquad\qquad \iota_j : S^{(j)}X \to SX$$

for any space X, sending the "algebraic barycenter" to the "topological barycenter" (obviously, it suffices to consider $j = 1$). Then $S^{(j)}(\mathcal{C}_k)$ is an operad, and although (for $X = \mathcal{C}_k$) (18) does not have an obvious homotopy inverse simplicial map of operads, after *simplicial realization*, there is a canonical map homotopy inverse. More precisely, if for the moment $|?|$ denotes topological simplicial realization, then

there is a canonical map of operads (by "triangulating singular simplices")

$$(19) \qquad |S\mathcal{C}_k| \to |S^{(j)}\mathcal{C}_k|$$

where the compositions of the simplicial realization of (18) and (19) both ways are homotopic to the identities through maps of operads. We see therefore that

$$\iota_j \square \iota_j : S^{(j)}\mathcal{C}_k \square S^{(j)}\mathcal{C}_\ell \to S\mathcal{C}_k \square S\mathcal{C}_\ell$$

is an equivalence (apply simplicial realization, commute $|?|$ past the box and use the homotopies cited above). Thus, we would be done if we could show that $\phi \circ (\iota_j \square \iota_j)$ applied to the n-th space of the source operads lands in $\mathcal{C}_m(n)'$ for some j. While this is obviously too much to expect, it is however true that for any *finite* simplicial subset T of $S\mathcal{C}_k \square S\mathcal{C}_\ell$ there exists a j such that the simplicial subset $T^{(j)}$ of

$$S^{(j)}\mathcal{C}_k \square S^{(j)}\mathcal{C}_\ell$$

consisting of j-fold barycentric subdivisions of simplices of T does have the property that

$$\phi \circ (\iota_j \square \iota_j)(T^{(j)}) \subset (S\mathcal{C}_m)(n)'.$$

By the Whitehead theorem, this is enough. $\qquad\square$

3. Lax algebras

This story is complex enough that it seems worth telling for the category of (simplicial) sets first. We use essentially the ideas of [14], [11], but as far as we know, they have not been recorded in this generality.

First, we recall the notion of theory in the sense of Lawvere [12]. Consider some type of universal algebra, given by certain number of n-ary operations (for various values of n), and relations which can be expressed by commutative diagrams. The *theory* $\mathcal{C}$ associated with the universal algebra type is then a sequence $\mathcal{C}(n)$ of sets, where $\mathcal{C}(n)$ consists of all different expressions one can write in n variables $x_1, ..., x_n$ (with repetitions allowed) where two expressions are considered equal if their equality follows from the relations of the universal algebra. A theory is in particular an operad, but has additional structure whose main feature is *substitution*, which can be expressed by saying that $\mathcal{C}$ is a functor $Map \to Sets$, where Map is the category whose objects are sets $\{1, ..., n\}$ and morphisms are maps of sets. Axioms relating composition and substitution arise from what is universally true in this setting. Associated with a theory $\mathcal{C}$ is a monad C in sets whose algebras are models of the given universal algebra type ($\mathcal{C}$-algebras). Monads associated with theories are called *finitary*.

Now for any theory $\mathcal{C}$, a *lax $\mathcal{C}$-algebra* is a category Cat where each n-ary operation μ of $\mathcal{C}$ corresponds to a functor

$$(20) \qquad \underline{\mu} : Cat^{\times n} \to Cat,$$

Further, for each $\mu \in \mathcal{C}(n)$, $\mu_i \in \mathcal{C}(n_i)$, $i = 1, ..., n$, we are required to have natural *coherence isomorphisms*

$$(21) \qquad \phi : \underline{\mu}(\underline{\mu}_1, ..., \underline{\mu}_n) \cong \underline{\gamma(\mu, \mu_1..., \mu_n)}, \ \phi : \underline{1} \to 1$$

where γ is the composition in $\mathcal{C}$, and also for $\kappa : \{1, ..., n\} \to \{1, ..., m\} \in Mor(Map)$,

$$(22) \qquad \phi : \underline{\kappa\mu} \cong \underline{\mu}\kappa^*$$

where $\kappa^* : Cat^{\times m} \to Cat^{\times n}$ is the categorically induced map by κ. In addition, the coherence isomorphisms (21), (22) are required to satisfy *coherence diagrams* expressing a cocycle condition for the isomorphisms (21). This means, for example, that for a 3-fold composition, applying the coherence isos in either order gives the same result:

$$\underline{\mu}(\underline{\mu}_1(\underline{\mu}_{11}, ..., \underline{\mu}_{1m_1}), ..., \underline{\mu}_n(\underline{\mu}_{n1}, ..., \underline{\mu}_{nm_n}))$$

$$\underline{\mu}(\gamma(\mu_1, \mu_{11}, ..., \mu_{1m_1}), ..., \underline{\gamma(\mu_n, \mu_{n1}, ..., \mu_{nm_n})})$$

$$\frac{\gamma(\mu, \mu_1, ..., \mu_n)}{(\underline{\mu}_1, ..., \underline{\mu}_{nm_n})}$$

$$\gamma(\mu, \gamma(\mu_1, \mu_{11}, ..., \mu_{1m_1}), ..., \gamma(\mu_n, \mu_{n1}, ..., \mu_{nm_n})).$$

There are also analogous diagrams involving unit and substitution: the general guideline is that theories can be considered as a kind of universal algebra whose operations O are composition, unit and substitution, and relations R are those identities which are true in all theories; then a lax algebra has coherence isomorphisms correspoinding to elements of O, and coherence diagrams corresponding to elements of R.

In order to give a complete and self-contained definition in this paper, we write down all the other coherence diagrams explicitly as well. These diagrams are also listed in [6].

The diagrams coming from theory unit axioms are

$$\underline{1}(\underline{\mu}) \xrightarrow{\cong \phi} \gamma(1, \underline{\mu})$$
$$\cong \phi(\underline{\mu}) \downarrow \qquad = \nearrow$$
$$\underline{\mu},$$

$$\underline{\mu}(\underline{1}, ..., \underline{1}) \xrightarrow{\cong \phi} \gamma(\mu, 1, ..., 1)$$
$$\underline{\mu}(\phi) \downarrow \qquad = \nearrow$$
$$\underline{\mu}.$$

Regarding substitutions, there are two diagrams (associativity and unit) coming from the axiom that $\mathcal{C} : Map \to Sets$ is a functor. If $\kappa : \{1, ..., n\} \to \{1, ..., m\}$, $\lambda : \{1, ..., m\} \to \{1, ..., \ell\}$ and $\iota = Id : \{1, ..., n\} \to \{1, ..., n\}$, the diagrams are

$$\begin{array}{ccc}
\underline{\lambda \kappa \mu} & \xrightarrow{\cong \phi} & \underline{\kappa \mu \lambda^*} \\
\cong \phi \downarrow & & \downarrow \cong \phi \lambda^* \\
\underline{\mu}(\lambda \kappa)^* & \xrightarrow{=} & \underline{\mu} \kappa^* \lambda^*,
\end{array}$$

$$\underline{\iota\mu} \xrightarrow{\cong \phi} \underline{\mu\iota}^*$$

$$= \Big\downarrow \qquad \nearrow \, =$$

$$\underline{\mu}.$$

Finally, one diagram comes from the commutation relation between composition and substitution in a theory. Let $\lambda : \{1, ..., n\} \rightarrow \{1, ..., m\}$, $\kappa_i : \{1, ..., k_i\} \rightarrow \{1, ..., \ell_i\}$, $i = 1, ..., m$. Then there is a map

$$\lambda \wr (\kappa_1, ..., \kappa_m) : \{1, ..., \sum_{i=1}^{n} k_{\lambda(i)}\} \rightarrow \{1, ..., \sum_{i=1}^{m} \ell_i\}$$

given by

$$(\lambda \wr (\kappa_1, ..., \kappa_m))(\sum_{i=1}^{j-1} k_{\lambda(i)} + p) = \sum_{i=1}^{\lambda(j)-1} \ell_i + \kappa_{\lambda(j)}(p)$$

for $j = 1, ..., n$, $p = 1, ...k_{\lambda(j)}$. The diagram then reads

$$
\begin{array}{ccc}
\underline{(\lambda \wr (\kappa_1, ..., \kappa_m))}\gamma(\mu, \mu_{\lambda(1)}, ..., \mu_{\lambda(n)}) & \xrightarrow{\cong \phi} & \underline{\gamma(\mu, \mu_{\lambda(1)}, ..., \mu_{\lambda(n)})}(\lambda \wr (\kappa_1, ..., \kappa_m))^* \\
\uparrow = & & \phi(\lambda\wr(\kappa_1,...,\kappa_m))^* \Big\uparrow \cong \\
\underline{\gamma(\lambda\mu, \kappa_1\mu_1, ...\kappa_m\mu_m)} & & \underline{\mu}(\underline{\mu}_{\lambda(1)}, ..., \underline{\mu}_{\lambda(n)})(\lambda \wr (\kappa_1, ..., \kappa_m))^* \\
\uparrow \cong \phi & & = \Big\uparrow \\
\underline{\lambda\mu}(\underline{\kappa_1\mu_1}, ..., \underline{\kappa_m\mu_m}) & \xrightarrow[\phi(\phi,...,\phi)]{\cong} & \underline{\mu}\lambda^*(\underline{\mu}_1\kappa_1^*, ..., \underline{\mu}_m\kappa_m^*).
\end{array}
$$

In another formulation, we may consider the free theory $Th(\mathcal{C})$ on $\mathcal{C}$ (i.e. the theory obtained by iterating the operations in $\mathcal{C}$, and performing substitutions and insertions of unit), and assign a canonical iso to any two elements in $Th(\mathcal{C})(n)$ whose images in $\mathcal{C}(n)$ coincide: these isos are required to be transitive and compatible with substitution and composition. Note that these isos, together with their iterations, now make each $Th(\mathcal{C})(n)$ into a groupoid $G(n)$, with trivial automorphism groups. The construction of $Th(\mathcal{C})$ is discussed in detail in Fiore [6]. Then $Th(\mathcal{C})$ defines a 2-monad in the sense of Blackwell, Kelly and Power [1]. Lax $\mathcal{C}$-algebras can then be identified with 2-algebras over the 2-monad $Th(\mathcal{C})$. Following [1], such 2-algebras form a 2-category where 1-morphisms are what [1] call pseudomorphisms.

Now let $\mathcal{C}$ be a theory. We define a simplicial theory $\overline{\mathcal{C}}$ as

$$\overline{C}(n) = BG(n) = B_{Obj(G(n))}(Obj(G(n)), Mor(G(n)), Obj(G(n)))$$
$$= B_{Th(C)(n)}(Th(C)(n), Mor(G(n)), Th(C)(n)).$$

We have a canonical projection

$$|\overline{\mathcal{C}}| \rightarrow \mathcal{C}$$

which is an equivalence since $G(n)$ has trivial automorphism groups and the image of every element in $\mathcal{C}$ is a connected groupoid.

Proposition 7. *Let Cat be a lax $\mathcal{C}$-algebra. Then $BCat$ is canonically a $\overline{\mathcal{C}}$-algebra.*

Proof: Let

$$A = \left(\begin{array}{ccc} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{array} \right)$$

be a matrix of morphisms in Cat, $Ta_{ij} = Sa_{i,j+1}$. (S, T mean source and target.)
Let $\phi = (\phi_1, ..., \phi_n)$ be a composable n-tuple in $G(m)$,

$$\phi_i : \sigma_{i-1} \to \sigma_i.$$

Then

$$\phi A = (\phi_1 \sigma_0 \left(\begin{array}{c} a_{11} \\ ... \\ a_{m1} \end{array} \right), \phi_2 \sigma_1 \left(\begin{array}{c} a_{12} \\ ... \\ a_{m2} \end{array} \right), ..., \phi_n \sigma_{n-1} \left(\begin{array}{c} a_{1n} \\ ... \\ a_{mn} \end{array} \right)).$$

$\square$

In an arbitrary symmetric monoidal ground category $\mathcal{B}$, we cannot speak of
theories with respect to the symmetric monoidal structure $\boxtimes$, since there is in
general no diagonal map

$$X \to X \boxtimes X.$$

However, we can speak of $\mathcal{C}$-algebra for an operad $\mathcal{C}$ (see also Ginzburg-Kapranov
[7]).

Specifically, recall that a *category C enriched in $\mathcal{B}$* consists of objects

$$Obj(C), Mor(C) \in Obj(\mathcal{B}),$$

and morphisms $S, T, Id, \gamma \in Mor(\mathcal{B})$, $S, T : Mor(C) \to Obj(C)$, $Id : Obj(C) \to Mor(C)$, $\gamma : Mor(C) \prod_{Obj(C)} Mor(C) \to Mor(C)$ (the source of γ denotes a pull-
back), with the usual axioms. We then have an object

$$BC \in Obj(s\mathcal{B})$$

defined by

$$BC_k = \underbrace{Mor(C) \prod_{Obj(C)} ... \prod_{Obj(C)} Mor(C)}_{k \text{ times}}.$$

Now let $\mathcal{C}$ be an operad in (simplicial) sets. There is a notion of $\mathcal{C}$-algebra X
enriched over $\mathcal{B}$. The structure maps are of the form

$$(23) \qquad\qquad \gamma : \mathcal{C}(n) \otimes (\underbrace{X \boxtimes ... \boxtimes X}_{n \text{ times}}) \to X$$

with the usual diagrams mimicking the diagrams defining an $\mathcal{C}$-algebra.

Now a *lax $\mathcal{C}$-algebra enriched over $\mathcal{B}$* is a category C enriched over $\mathcal{B}$ together
with functorial structure maps (23) for $X = Obj(C), Mor(C)$ and coherence iso
structure of the following form:

For $x \in \mathcal{C}(n)$, $y_i \in \mathcal{C}(k_i)$, $i = 1, ..., n$,

$$\phi_{x,y_1,...,y_n} \in Mor(\mathcal{B}) : Obj(C)^{\boxtimes \sum k_i} \to Mor(C)$$

with a commutative diagram

$$
\begin{array}{ccc}
Obj(C)^{\boxtimes n} & \xleftarrow{\;\boxtimes \gamma_{y_i}\;} & Obj(C)^{\boxtimes \sum k_i} \\[2pt]
{\scriptstyle \gamma_x}\Big\downarrow & & \Big\downarrow{\scriptstyle \gamma_{\gamma(x,y_1,\ldots,y_n)}} \\[2pt]
Obj(C) & \xrightarrow{\;\phi_{x,y_1,\ldots,y_n}\;} & Obj(C)
\end{array}
$$

and corresponding coherence diagrams, mimicking the coherence diagrams in the case of sets. One also must not forget to include coherence isomorphisms corresponding to substitutions by *permutations* (the Σ_n-action on the n-th space of an operad), and coherence diagrams corresponding to axioms involving composition and equivariance (cf. [14]). The fact that "the targets of $\phi_{x,y_1,\ldots,y_n}$ are iso" is expressed, for example, by giving an 'inverse' map

$$
\psi_{x,y_1,\ldots,y_n} \in Mor(\mathcal{B}) : Obj(C)^{\boxtimes \sum k_i} \to Mor(C).
$$

The same method (with the groupoid theory $Th(\mathcal{C}), G(\mathcal{C})$ replaced by the free operad $Op(\mathcal{C})$ on $\mathcal{C}$ which is made into a groupoid by using as morphisms all the operad coherence isomorphisms $H(\mathcal{C})$) then gives a simplicial operad $\overline{\mathcal{C}}$ and a map of operads which is an equivalence in $s\mathcal{B}$

$$
|\overline{\mathcal{C}}| \to \mathcal{C}.
$$

(This map is an equivalence because the inverse image of every word in $\mathcal{C}$ is a connected groupoid with no non-identical automorphisms.) Thus, we get

Proposition 8. *Let Cat be a lax $\mathcal{C}$-algebra enriched over $\mathcal{B}$. Then $BCat$ is canonically a $\overline{\mathcal{C}}$-algebra in $\mathcal{B}$.*

$$\square$$

4. Lax algebras enriched over categories

We shall now need to consider an even further generalization. Let us, again, first work in the context of simplicial sets, where the structure is simpler than in the $\mathcal{B}$-enriched case. In this case, the appropriate notion is a lax $\mathcal{C}$-algebra Cat enriched over categories. This means that Cat has a structure analogous to that of lax $\mathcal{C}$-algebra (to fix ideas, let $\mathcal{C}$ be an operad, but the case of theories is analogous), where both $Obj(Cat)$, $Mor(Cat)$ are categories (which we will refer to as the *vertical* categories), and all structure maps of lax $\mathcal{C}$-algebra are functors (and coherence diagrams commute strictly, rather than just up to natural isomorphisms). Therefore, in addition to the vertical categories, we get two lax $\mathcal{C}$-algebras

$$
(24) \qquad\qquad\qquad (Obj_{Obj(Cat)}, Obj_{Mor(Cat)})
$$

and

$$
(25) \qquad\qquad\qquad (Mor_{Obj(Cat)}, Mor_{Mor(Cat)}).
$$

(We use the subscript notation to distinguish this structure from the vertical categories.) In particular, then, (24), (25) are categories, and we will refer to them

as the *horizontal* categories. To spell out our notation completely, the vertical categories then are

$$Obj(Cat) = (Obj_{Obj(Cat)}, Mor_{Obj(Cat)}),$$
$$Mor(Cat) = (Obj_{Mor(Cat)}, Mor_{Mor(Cat)}).$$

Now suppose we are given a lax $\mathcal{C}$-algebra Cat enriched over categories. Then performing the horizontal bar construction (which we will denote by B_{hor}), we obtain a strict simplicial $\overline{\mathcal{C}}$-algebra enriched over categories (using the vertical structure):

$$B_{hor}(Cat) = \begin{pmatrix} B_{hor}(Obj_{Cat}) \\ B_{hor}(Mor_{Cat}) \end{pmatrix}.$$

Observe that $B_{hor}(Cat)$ is "almost" a $\mathcal{C}\square\mathcal{C}_1$-algebra: the $\mathcal{C}$-algebra structure was just described, and the "$\mathcal{C}_1$-structure" can be pulled back from the (associative) categorical composition. The difficulty with that is that the source of the categorical composition is not the product

(26) $$B_{hor}(Mor_{Cat}) \times B_{hor}(Mor_{Cat}),$$

but the fibered product

(27) $$B_{hor}(Mor_{Cat}) \times_{B_{hor}(Obj_{Cat})} B_{hor}(Mor_{Cat}).$$

There is a natural inclusion of (27) to (26). We need a technique for extending the domain of the composition product from (27) to (26).

The technique we shall use is the two-sided bar construction of monads [14]. The ground category is the category $\mathcal{G}$ of *graphs* of $\overline{\mathcal{C}}$-algebras over $B = B_{hor}(Obj_{Cat})$, which means $\overline{\mathcal{C}}$-algebras X with two maps of $\overline{\mathcal{C}}$-algebras

$$S, T : X \to B.$$

Then the monad in $\mathcal{G}$ which defines categories in $\overline{\mathcal{C}}$ with objects B (where, as above, composition commutes with $\overline{\mathcal{C}}$-algebra structure) is

(28) $$DX = \coprod_{n \geq 0} \underbrace{X \times_B ... \times_B X}_{n \text{ times}}.$$

On the other hand, the monad which defines monoids in $\overline{\mathcal{C}}$-algebras (i.e. again, we require that the $\overline{\mathcal{C}}$-algebra structure commutes with composition) is

(29) $$EX = \coprod_{n \geq 0} \underbrace{X \times ... \times X}_{n \text{ times}}.$$

Clearly, we have a map of monads $D \to E$, and we can therefore consider the 2-sided bar construction of monads

(30) $$B(E, D, X).$$

Then (the realization of) (30) is a monoid in $\overline{\mathcal{C}}$-algebras, i.e. a $\overline{\mathcal{C}}\square\mathcal{C}_1$-algebra by pullback. Of course, we would like to compare the homotopy type of (30) to the homotopy type of X. As usual, we have the comparison map

(31) $$X \xleftarrow{\;\simeq\;} B(D, D, X) \longrightarrow B(E, D, X).$$

When is the second map (31) an equivalence? An obvious condition is

(32) $$B = B_{hor}(Obj_{Cat}) \text{ is contractible.}$$

However, (32) per se unfortunately does not suffice, as we need some local condition. Lemma 7.2 of [14] implies that for example the following condition suffices. In order to formulate the condition, we must slightly change our context: we shall actually assume that (24), (25) are *simplicial* categories where the simplicial structure of (25) is constant (we will see in the next section that such situation arises naturally). We may of course always realize to make objects simplicial, but for the purposes of the following condition, X, B are then bisimplicial sets, where in one of the simplicial directions B is constant. The condition reads as follows:

(33)   Considering the projection map $S \times T : X \to B \times B$, then for every $b \in (B \times B)_n$, and every face d_i (resp. s_i, the face (degeneracy) $(S \times T)^{-1}b \to (S \times T)^{-1}d_ib$, $(S \times T)^{-1}b \to (S \times T)^{-1}s_ib$ are equivalences.

Then every projection map

$$(34) \qquad \underbrace{X \times_B \dots \times_B X}_{n \text{ times}} \to B^{\times(n+1)}$$

is a quasifibration, so if F is any fiber of (34), we have a diagram

and hence the vertical map is an equivalence. It follows that the right hand map (31) is an equivalence. Note also that this implies that X is equivalent, via inclusion, to any fiber of any of the maps S, T.

Now we need to translate (33) to some condition on categories which could be applied in the case when $X = B_{hor}(Mor_{Cat})$. The following condition is obviously sufficient (Hom_{vert} denotes vertical Hom sets):

(35)   Let $M, N, M', N' \in Obj_{Obj(Cat)}$, let $f : M \to M'$, $g : N \to N'$ be horizontal morphisms. Then the natural projections $S : Hom_{vert}(f, g) \to Hom_{vert}(M, N)$, $T : Hom_{vert}(f, g) \to Hom_{vert}(M', N')$ are equivalences, and S is a fibration.

We shall call *Cat distinguished* if (32), (35) are satisfied. We therefore have proven

Proposition 9. *Let Cat be a distinguished lax $\mathcal{C}$-algebra enriched over categories. Then for each $M, N \in Obj_{Obj(Cat)}$, $Hom_{vert}(M, N)$ are naturally equivalent to each other, and moreover naturally equivalent to a $\overline{\mathcal{C}} \square \mathcal{C}_1$-algebra.*

Remark: Recall that May's two-sided bar construction of monads [14] allows us, for any map of operads $\mathcal{D}_1 \to \mathcal{D}_2$ which is an equivalence, and for any $\mathcal{D}_1$-algebra X, to construct the equivalent $\mathcal{D}_2$-algebra

$$B(D_2, D_1, X)$$

where D_i are the monads associated with the operads $\mathcal{D}_i$. Therefore, in Proposition 9, we are allowed to replace $\overline{\mathcal{C}} \square \mathcal{C}_1$ with $\mathcal{C} \square \mathcal{C}_1$ or any other equivalent operad desired.

Now to treat the case enriched over a symmetric monoidal category $\mathcal{B}$, we start by defining a *category Cat enriched in $\mathcal{B}$-categories*. Such structure consists of the following data: First, we have a 'horizontal object category'

$$Obj_{Cat} = (Obj_{Obj(Cat)}, Obj_{Mor(Cat)})$$

which is an ordinary category (i.e. "enriched" only over Sets).

Next, we have a 'vertical object category' given by specifying, for each $x, y \in Obj_{Obj(Cat)}$, a

$$Hom(x, y) \in Obj(\mathcal{B}).$$

Further, there are specified maps

$$Id \in Mor(\mathcal{B}) : 1_\square \to Hom(x, x),$$

$$\gamma \in Mor(\mathcal{B}) : Hom(x, y) \boxtimes Hom(y, z) \to Hom(x, z)$$

with usual axioms of associativity and unity and a 'vertical morphism category' specifying, similarly, for $f, g \in Obj_{Mor(Cat)}$, a

$$Hom(f, g) \in Obj(\mathcal{B}).$$

Further, there are specified maps

$$Id \in Mor(\mathcal{B}) : 1_\square \to Hom(f, f),$$

$$\gamma \in Mor(\mathcal{B}) : Hom(f, g) \boxtimes Hom(g, h) \to Hom(f, h)$$

with usual axioms of associativity and unity.

Next, we have a 'horizontal morphism category' Mor_{Cat} enriched over $\mathcal{B}$: For

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \ ,$$

$$y_0 \xrightarrow{g_1} y_1 \xrightarrow{g_2} y_2$$

in Obj_{Cat}, morphisms $T \in Mor(\mathcal{B}) : Hom(f_1, g_1) \to Hom(x_1, y_1)$, $S \in Mor(\mathcal{B}) : Hom(f_2, g_2) \to Hom(x_1, y_1)$ and

$$\gamma \in Mor(\mathcal{B}) : Hom(f_1, g_1) \prod_{Hom(x_1, y_1)} Hom(f_2, g_2) \to Hom(f_2 f_1, g_2, g_1)$$

and also

$$Id \in Mor(\mathcal{B}) : Hom(x, x) \to Hom(Id_x, Id_x).$$

Finally, there is a diagram of commutativity between vertical and horizontal composition. This diagram expresses the equality of two maps from

$$(36) \quad (Hom(f_1, g_1) \prod_{Hom(x_1, y_1)} Hom(f_2, g_2)) \boxtimes (Hom(g_1, h_1) \prod_{Hom(y_1, z_1)} Hom(g_2, h_2))$$

to

$$(37) \qquad\qquad\qquad Hom(f_2 f_1, h_2 h_1).$$

The first map maps (36) to

$$Hom(f_2 f_1, g_2 g_1) \boxtimes Hom(g_2 g_1, h_2 h_1)$$

by horizontal composition, and then maps to (37) by vertical composition. The second map maps (36) to

$$(Hom(f_1, g_1) \boxtimes Hom(g_1, h_1)) \prod_{(Hom(x_1, y_1) \boxtimes Hom(y_1, z_1))} (Hom(f_2, g_2) \boxtimes Hom(g_2, h_2))$$

using the limit properties of a pullback, followed by a map to

$$Hom(f_1, h_1) \prod_{Hom(x_1, z_1)} Hom(f_2, h_2)$$

by vertical composition, and then to (37) by horizontal composition.

Now the axioms of a *lax $\mathcal{C}$-algebra Cat enriched in $\mathcal{B}$-categories* consists of the following data:

  (1) An ordinary lax $\mathcal{C}$-algebra structure on Obj_{Cat}.
  (2) A structure of lax $\mathcal{C}$-algebra enriched in $\mathcal{B}$ on Mor_{Cat}, compatible with the fibering of Mor_{Cat} over Obj_{Cat}.
  (3) Compatibility diagrams of (2) with vertical unit and composition.

Similarly as before, we can also make Cat a simplicial object in the kind of structures just described, and for the homotopical part of our discussion we will find it advantageous to also assume that, in addition with the horizontal category Obj_{Cat} being simplicially constant.

Now to obtain an analogue of Proposition 9 for lax algebras enriched over $\mathcal{B}$-categories, we will examine the construction leading up to Proposition 9, noting along the way how they must be changed in view of $\mathcal{B}$-enrichment.

First, we examine $B_{hor}(Cat)$. We see that $B_{hor}(Obj_{Cat})$ is a $\overline{\mathcal{C}}$-algebra over sets, and $B_{hor}(Mor_{Cat})$ is a $\overline{\mathcal{C}}$-algebra enriched over $\mathcal{B}$ *fibered* over $B = B_{hor}(Obj_{Cat})$. We further have associative composition, which is a map of $\overline{\mathcal{C}}$-algebras,

$$B_{hor}(Mor_{Cat}) \boxtimes_B B_{hor}(Mor_{Cat}) \to B_{hor}(Mor_{Cat})$$

(the symbol $\boxtimes_B$ indicates $\boxtimes$ applied fiber-wise). Again, what we want is to be able to extend the product in a way so that we can replace $\boxtimes_B$ with $\boxtimes$. Again, we have a category of $\mathcal{B}$-enriched $\overline{\mathcal{C}}$-algebras fibered over B, and a map of monads $D \to E$ in this category,

$$DX = \coprod_{n \geq 0} \underbrace{X \boxtimes_B \dots \boxtimes_B X}_{n \text{ times}},$$

$$EX = \coprod_{n \geq 0} \underbrace{X \boxtimes \dots \boxtimes X}_{n \text{ times}},$$

may then form the two-sided bar construction of monads (30) and note that (30) is a $\mathcal{B}$-enriched $\overline{\mathcal{C}} \square \mathcal{C}_1$-algebra. Therefore, we must again find conditions when the second map (31) is an equivalence, when $X = B_{hor}(Mor_{Cat})$. The conditions we arrive at are again (32) and (35), although while (32) does not change, in (35) the Hom_{vert} sets now denote objects of $s\mathcal{B}$. When these conditions are satisfied, we say, again, that Cat is *distinguished*. We therefore have an enriched analogue of Proposition 9:

Proposition 10. *Let Cat be a distinguished lax $\mathcal{C}$-algebra enriched over $\mathcal{B}$-categories. Then for $M, N \in Obj_{Obj(Cat)}$, $Hom_{vert}(M, N)$ are all naturally equivalent, and naturally equivalent to a $\overline{\mathcal{C}} \square \mathcal{C}_1$-algebra enriched over $\mathcal{B}$.*

5. Proof of Theorem 3

In view of Proposition 10, it suffices to produce a distinguished lax $\mathcal{C}$-algebra Cat enriched in $\mathcal{B}$-categories, where for $M, N \in Obj_{Obj(Cat)}$,

$$Hom_{vert}(M, N)$$

is naturally equivalent to (11).

Let Q be a fibrant special operad in $s\mathcal{B}$ fibered over $\mathcal{C}$, a cofibrant operad in simplicial sets with $\mathcal{C}(0) = *$, $\mathcal{C}(1) \simeq *$. Define the category Cat enriched in $s\mathcal{B}$-categories as follows: If $\mathcal{B} = k$-modules, the horizontal category Obj_{Cat} has as objects all fibration equivalences of Q_1-modules

$$(38) \qquad\qquad\qquad\qquad M \to Q(0)$$

where M is cofibrant (note that $Q(0)$ is fibrant). If $\mathcal{B} = Sets$, drop the requirement that (38) be a fibration. The (horizontal) morphisms in Obj_{Cat} are commutative diagrams

$$(39)$$

$$
\begin{array}{ccc}
M & \longrightarrow & Q(0) \\
{\scriptstyle f}\downarrow & \nearrow & \\
N & &
\end{array}
$$

where the map $f : M \to N$ is a cofibration. The vertical Hom-sets in $Obj(Cat)$ (i.e. between two objects of the form (38)) are

$$Hom_{Q_1}(M, M'),$$

if $\mathcal{B} = k$-modules.

When $\mathcal{B} = Sets$, we could actually use the *space*

$$(40) \qquad\qquad\qquad\qquad Map_{|Q_1|}(|M|, |M'|).$$

However, since we want to stay in the category of simplicial sets, we get back by applying the singular set functor to (40).

The vertical Hom-sets in $Mor(Cat)$ are groups of pairs (under addition) of

$$(u, v) \in Hom_{Q_1}(M, N) \prod Hom_{Q_1}(M', N')$$

for K-modules and

$$(u, v) \in Map_{|Q_1|}(|M|, |N|) \prod Map_{|Q_1|}(|M'|, |N'|)$$

for sets which make commutative the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{f} & M' \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\
N & \xrightarrow{g} & N'.
\end{array}
$$

Note that the requirement that f be a cofibration implies (for K-modules) that the map induced by f

$$Hom(M', N') \to Hom(M, N')$$

is a fibration equivalence. Therefore

$$Hom(f, g) = Hom(M, N) \prod_{Hom(M,N')} Hom(M', N')$$

$$\simeq \Big\downarrow$$

$$Hom(M, N)$$

is a fibration equivalence. Now we have a diagram (where the arrows are induced maps)

$$
\begin{array}{ccc}
Hom(f, g) & \longrightarrow & Hom(M', N') \\
\simeq \Big\downarrow & & \Big\downarrow \simeq \\
Hom(M, N) & \underset{\simeq}{\longrightarrow} & Hom(M, N').
\end{array}
$$

The bottom row is an equivalence because $N \to N'$ is an equivalence, and by our hypothesis, N, N' are fibrant. Similarly for sets. This already proves condition (35).

Now Q being special implies that Cat is a lax $\mathcal{C}$-algebra enriched over $s\mathcal{B}$-categories. In effect, the lax $\mathcal{C}$-algebra structure is defined as follows: for $x \in \mathcal{C}(n)$, and Q_1-modules $M_1, ..., M_n$, the x-product of $M_1,...,M_n$ is

$$(41) \qquad Q(n)_x \boxtimes_{Q_1^{\boxtimes n}} (M_1 \boxtimes ... \boxtimes M_n).$$

If Q is special, cofibrant and fibrant, the canonical map from (41) to $Q(0)$ is an equivalence (see (7)). On morphisms, (41) preserves Q_1-cofibrations if Q is cofibrant. For $\mathcal{B} = K$-modules, (41) preserves fibrations, because fibrations of simplicial K-modules are precisely onto maps. Note that this operation is functorial, and on morphisms carries cofibrations to cofibrations, with $s\mathcal{B}$ as above.

Thus, the proof of the statement that Cat is distinguished (and hence of Theorem 3) is reduced to the following

Proposition 11.

$$(42) \qquad BObj_{Cat} \simeq *.$$

Proof: To show (42), one chooses a particular object

$$M_0 \to Q(0).$$

For any object of Obj_{Cat}

$$M \to Q(0),$$

one can obviously choose an arrow $N_M \to Q(0)$ in Obj_{Cat} together with a diagram in Obj_{Cat}

$$
\begin{array}{ccccc}
M & \longrightarrow & N_M & \longleftarrow & M_0 \\
& \simeq \searrow & \Big\downarrow \simeq & \swarrow \simeq & \\
& & Q(0). & &
\end{array}
$$

Similarly, for a (horizontal) morphism in Obj_{Cat}

$$M \xrightarrow{\quad f \quad} M'$$
$$\simeq \searrow \qquad \swarrow \simeq$$
$$Q(0),$$

there is an $N_f \to Q(0)$ in Obj_{Cat} such that, over $Q(0)$, we have a commutative diagram

$$
\begin{array}{ccccc}
M & \longrightarrow & N_M & \longleftarrow & M_0 \\
& & & & \\
& & N_f & & = \\
& & & & \\
M' & \longrightarrow & N_{M'} & \longleftarrow & M_0.
\end{array}
$$

To formalize the procedure this will give, recall the *barycentric subdivision* C' of a category C (in our case, $C = Obj_{Cat}$): The category C' is a partially ordered set whose objects are n-tuples of composable arrows in C

$$(43) \qquad x_0 \xrightarrow{\ f_1\ } x_1 \longrightarrow \cdots \xrightarrow{\ f_n\ } x_n,$$

and a k-tuple of composable morphisms

$$\cdot \xrightarrow{\ g_1\ } \cdot \longrightarrow \cdots \xrightarrow{\ g_k\ } \cdot$$

is said to be $\leq$ (43) if the g_i's are obtained by consecutive compositions (or omissions from the beginning or end) of the f_i's. Then there is a canonical functor

$$\Gamma : C' \to C$$

which on objects is given by

$$(f_n, ..., f_1) \mapsto Tf_n$$

(and by the obvious formula on morphisms). The functor Γ induces an equivalence upon applying B (the bar construction).

Now repeating the procedure we described constructs a functor

$$N : (Obj_{Cat})' \to Obj_{Cat},$$

together with natural transformations

$$(44) \qquad\qquad\qquad\qquad \Gamma \to N,$$

$$(45) \qquad\qquad\qquad\qquad G \to N$$

where G is the constant functor with value in $M_0 \to Q(0)$, thus showing that $BObj_{Cat}$ is contractible, as claimed. $\qquad\square$

6. The special property for k-cubes

The purpose of this section is to prove Theorem 6. In this section $\mathcal{C}_k$ will stand for the topological version of the little k-cube operad, and we will also consider the topological version of the construction (12), $\mathcal{C} = \mathcal{C}_k$. Recall that $D_0(X) = C_k X$. It clearly suffices to prove that the map

$$(46) \qquad \phi : B(D_n X, (D_1 X)^{\times n}, (D_0 X)^{\times n}) \to CX \times B(\mathcal{C}_k(n), \mathcal{C}_k(1)^{\times n}, \mathcal{C}_k(0)^{\times n})$$

is an equivalence, where the first coordinate of the map ϕ is given by composition, and the second map by the forgetful map

$$D_\ell(X) \to \mathcal{C}_k(\ell).$$

In effect, to get from spaces to simplicial sets, we may apply the singular set functor, and to get to K-modules, we may further apply the free K-module functor.

Now since the left hand side of (46) obviously preserves weak equivalences, we can further replace the terms of (46) as follows. First, let $\mathcal{C}'_k(\ell) \in \mathcal{C}_k(\ell)$ consist of all ℓ-tuples of little cubes $\alpha_1, ... \alpha_\ell : I^k \to I^k$ such that $Im(\alpha_i) \subset Int(I^k)$ and the images $Im(\alpha_i)$ are disjoint. Now let $M(\ell)$ be the space of pairs (α, m) where $\alpha = (\alpha_1, ..., \alpha_\ell) \in \mathcal{C}_k(\ell)$ and m is a set of unordered X-decorated points in $I^k - \bigcup_{i=1}^{\ell} Im(\alpha_i)$ (with the usual configuration space topology). Let $M'(\ell)$ be defined in the same way as $M(\ell)$, with $\mathcal{C}_k(\ell)$ replaced by $\mathcal{C}'_k(\ell)$:

$$M'(\ell) = M(\ell) \times_{\mathcal{C}_k(\ell)} \mathcal{C}'_k(\ell).$$

Then we have equivalences

$$D_n(X) \to M(n) \leftarrow M'(n)$$

(the first map replaces a little cube decorated by an element of X by its center). Thus, we can further restate the claim that (46) is an equivalence as follows:

Proposition 12. *The map*

$$(47) \qquad \kappa : B(M'(\ell), M(1)^{\times \ell}, M(0)^{\times \ell}) \to M(0) \times B(\mathcal{C}'_k(\ell), \mathcal{C}_k(1)^{\times \ell}, \mathcal{C}_k(0)^{\times \ell})$$

where the first map is by composition and the second map is by projection $M \to \mathcal{C}$, is an equivalence.

Remark: Recall that $\mathcal{C}'_k(\ell) \simeq \mathcal{C}_k(\ell)$, $\mathcal{C}_k(1) \simeq *$, so the right hand side of (47) is weakly equivalent to $M(0) \times \mathcal{C}_k(\ell)$.

Proof of Proposition 12: First define $N(\ell)$ as the pullback

$$(48) \qquad\qquad N(\ell) = (\mathcal{C}'_k(\ell) \times \mathcal{C}_k(1)^\ell) \times_{\mathcal{C}'_k(\ell)} M'(\ell)$$

where the map

$$\mathcal{C}'_k(\ell) \times \mathcal{C}_k(1)^\ell \to \mathcal{C}'_k(\ell)$$

is by composition. Then $N(\ell)$ enjoys a right $M(1)^\ell$-action where $M(1)^\ell$ acts by composition on $M'(\ell)$ and by the forgetful map together with internal composition

on $\mathcal{C}'_k(\ell) \times \mathcal{C}_k(1)^\ell$:

$$(\mathcal{C}'_k(\ell) \times \mathcal{C}_k(1)^\ell) \times M(1)^\ell \longrightarrow \mathcal{C}'_k(\ell) \times \mathcal{C}_k(1)^\ell \times \mathcal{C}_k(1)^\ell$$
$$\downarrow {\scriptstyle Id \times \gamma^\ell}$$
$$\mathcal{C}'_k(\ell) \times \mathcal{C}_k(1)^\ell.$$

Furthermore, the projection

$$N(\ell) \to M'(\ell)$$

is obviously an equivalence (by contracting $\mathcal{C}_k(1)^\ell$), so we may replace $M'(\ell)$ by $N(\ell)$ in the statement of the Proposition.

Now filter $N(\ell)$ by closed subspaces $N_q(\ell)$ consisting of all triples

$$(49) \qquad\qquad (\alpha, \beta, x),$$

$\alpha = (\alpha_1, ..., \alpha_\ell) \in \mathcal{C}'_k(\ell)$, $\beta = (\beta_1, ..., \beta_\ell) \in \mathcal{C}'_k(1)^\ell$, $x \in M'(\ell)$ where the number of X-decorated points of x contained n

$$I^k - \bigcup_{i=1}^{\ell} Im(\alpha_i)$$

is $\leq q$.

Lemma 13. *Suppose $A \subset X$ is an M-equivariant weak NDR pair with Urysohn function and homotopy (u, h) where M is a monoid acting on the right, such that the following conditions are satisfied:*

(1) $X - A = V \times M$ for some $V \subset X - A$ closed.
(2) $h_t(A \cup V) \subset A \cup V$ for all t. (Note that $A \cup V = A \cup Cl(V)$.)

Then there is a natural M-equivariant homotopy equivalence

$$Cofiber(A \subset X) \to ((A \cup V)/A) \wedge M_+.$$

Proof: Because of the weak NDR property, we have

$$Cofiber(A \subset X) \simeq X/A,$$

so it suffices to prove

$$(50) \qquad\qquad X/A \simeq (A \cup V)/A \wedge M_+.$$

To get a map $\leftarrow$, note that the inclusion

$$A \cup V \to X$$

extends to an equivariant map

$$(A \cup V) \times M \to X \to X/A$$

which clearly annihilates $A \times M$, thus inducing

$$\psi (A \cup V) \times M/A \times M \to X/A.$$

To get a map $\to$ in (50), use the map ϕ induced by h_1: to see that it is continuous, let

$$U = \{x \in X | u(x) < 1\}.$$

Then ϕ is constant (hence continuous) on U, but also continuous on $X - A$. Now $\{U, X - A\}$ is an open cover of X.

Now since ψ is obviously a bijection, we can define both homotopies $\phi\psi \simeq Id$, $\psi\phi \simeq Id$ as h_t. Then $\psi\phi$ is a quotient of h_t (with topology), and thus is continuous. To see that

$$(51) \qquad h_t : \phi\psi \simeq Id$$

is continuous, note that we have a continuous map

$$h_t : A \cup V \to A \cup V,$$

and hence

$$(52) \qquad unit \circ h_t : A \cup V \to (A \cup V) \wedge M_+.$$

Then (51) is the free extension of (52). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now below we shall prove

Lemma 14. *The pair $N_{q-1}(\ell) \subset N_q(\ell)$, with $M = M(1)^\ell$, satisfy the hypotheses of Lemma 13.*

Thus, so do $B(N_{q-1}(\ell), M(1)^\ell, M(1)^\ell) \subset B(N_q(\ell), M(1)^\ell, M(1)^\ell)$.
Now consider the diagram

$$(53)$$

$$
\begin{array}{ccc}
B(N_{q-1}(\ell), M(1)^\ell, M(0)^\ell) & \longrightarrow & B(N_q(\ell), M(1)^\ell, M(0)^\ell) \\
\downarrow & & \downarrow \\
N_{q-1}(\ell) \times_{M(1)^\ell} M(0)\ell & \longrightarrow & N_q(\ell) \times_{M(1)^\ell} M(0)^\ell.
\end{array}
$$

By Lemma 13, the cofibers of both rows are naturally equivalent to

$$((A \cup V)/A) \wedge M(0)^\ell_+.$$

Thus, if the left column of (53) is an equivalence, so is the right column. Thus, inductively,

$$B(N(\ell), M(1)^\ell, M(0)^\ell) \simeq N(\ell) \times_{M(1)^\ell} M(0)^\ell \cong \mathcal{C}'_k(\ell) \times M(0).$$

The equivalences are easily checked to be compatible with the required maps. $\quad\square$

Proof of Lemma 14: To simplify notation, we shall identify little cubes with their images. For a little cube α in I^k, and for $t \in \mathbb{R}_{>0}$, let $t\alpha$ be the cube with the same center which is t times larger.

Now first note that there is a continuous function

$$\lambda : \mathcal{C}'_k(\ell) \to \mathbb{R}_{>1}$$

such that for $\alpha = (\alpha_1, ..., \alpha_\ell) \in \mathcal{C}'_k(\ell)$, $\lambda = \lambda(\alpha)$,

$$(54) \qquad (\lambda\alpha_1, ..., \lambda\alpha_\ell) \in \mathcal{C}'_k(\ell).$$

To this end, for every $\beta \in \mathcal{C}'_k(\ell)$, there is an open neighbourhood U of β and a *constant* $\mu > 1$ which works as λ in (54) for $\alpha \in U$. Thus, since $\mathcal{C}'_k(\ell)$ is paracompact, the function λ can constructed by partition of unity.

Now also note that for two pairs of cubes $\alpha \subset \beta$, $\gamma \subset \delta$ (not equal), there is a canonical homeomorphism

$$\Phi^{\alpha\beta}_{\gamma\delta} Cl(\beta - \alpha) \to Cl(\delta - \gamma).$$

First, there are canonical linear homeomorphisms identifying the boundaries $\partial\alpha, \partial\beta, \partial\gamma, \partial\delta$. Connecting two corresponding points on $\partial\alpha$, $\partial\beta$ creates a line segment; map this segment linearly onto the line segment obtained by identifying the corresponding points on $\partial\gamma$, $\partial\delta$. Let also $(\Phi_{\gamma\delta}^{\alpha\beta})_*$ be the map induced by $\Phi_{\gamma\delta}^{\alpha\beta}$ on configuration spaces.

Then define the homotopy h in $N_q(\ell)$ by

$$h_t(\alpha, 1, m) = (((1 - t/3)\alpha_i)_i, 1, ((\Phi_{\lambda(\alpha)\alpha_i,(1-t/3)\alpha_i}^{\lambda(\alpha)\alpha_i,\alpha_i})_i)_* m),$$

and extend $M(1)^\ell$-equivariantly. We can let

$$V$$

consist of all triples $(\alpha, 1, m)$ where m contains exactly q X-decorated points. Define

$$u_t(\alpha, \beta, m) = sup \left\{ x \left| \begin{array}{l} \text{there are } \leq \ q \ X\text{-decorated points in} \\ ((\Phi_{\lambda(\alpha)\alpha_i,(1-t/3)\alpha_i}^{\lambda(\alpha)\alpha_i,\alpha_i})_i)_* m \text{ which are in } I^k - \\ \bigcup_{i=1}^{\ell} \alpha_i. \end{array} \right. \right\}$$

$\square$

7. Discussion and Concluding Remarks

We shall make some remarks about the relationship of our results with Kontsevich's paper [9]. Let, in this section, $\mathcal{B}$ be the category of chain complexes of K-modules.

Kontsevich conjectures (and proves in some cases) a relationship between the "deformation complex" and a "Hochschild complex" of a $\mathcal{C}$-algebra R. His deformation complex is really Quillen cohomology in the category of $\mathcal{C}$-algebras over R, i.e. maps of $\mathcal{C}$-algebras of the form

$$? \to R.$$

The underlying story here is Koszul duality.

For any category C (with, say, a suitable Quillen model structure with simplicial realization), Quillen [16] defines homology as left derived functor of abelianization. Here abelianization is the left adjoint L to the forgetful functor $R : Ab(C) \to C$ where $Ab(C)$ is the category of abelian group objects in C, with respect to the categorical product. (Note that one can alternately also consider the category $E_\infty C$ of E_∞ objects, i.e. objects with an action of an E_∞ operad, again with respect to the categorical product in C.)

The *Koszul transform* $C_!$ of the category C is then the category of coalgebras over the comonad LR in the category $Ab(C)$ (or alternately $E_\infty C$ - we shall stick to $Ab(C)$, and assume, for the purposes of this discussion, that K is of characteristic 0). The Koszul dual category $C^!$ is the opposite category to the Koszul transform $C_!$. There is an obvious functor from $C \to C_!$ (or contravariant funtor from C to $C^!$) which assigns $X \mapsto LX$. The corresponding functors on derived categories may be denoted by $X \mapsto X_!$, $X \mapsto X^!$, and called the Koszul transform and Koszul dual

of X. There are comparison maps relating $C^{!!}$ to C and $X^{!!}$ to X, but in general convergence issues stand in the way of asserting an equivalence of categories.

Koszul duality was worked out in detail by Ginzburg and Kapranov [7] for the categories of $\mathcal{C}$-algebras (with respect to $\otimes$) in the category of chain complexes of K-modules (at least when K has characteristic 0, and $\mathcal{C}$ is based, i.e., say, $\mathcal{C}(0) = K$. We should mention that the setting of [7] differs slightly from ours in that they consider operads not in simplicial sets, but internally in the category of chain complexes, with respect to $\otimes$.) In this case, the category of $\mathcal{C}$-algebras is *based* in the sense that it has the same initial and terminal object, namely K.

In this case, the (derived) Koszul dual of $\mathcal{C}$-algebras is the category of $\mathcal{C}^!$-algebras, where $\mathcal{C}^!$ is the free operad on $\mathcal{C}^\vee$ ($?^\vee$ denotes $Hom_k(?, k)$) with differentials coming from the operad structure on $\mathcal{C}$. The Koszul dual $R^!$ of a $\mathcal{C}$-algebra R is the free $\mathcal{C}^!$-algebra on $R^\vee$, with differentials coming from the $\mathcal{C}$-algebra structure on R. Further, the Koszul dual of the category of $(\mathcal{C}, R)$-modules is the category of $(\mathcal{C}^!, R^!)$-modules, and for a $(\mathcal{C}, R)$-module M, the Koszul dual $M^!$ is the free $(\mathcal{C}^!, R^!)$-module on $M^\vee$ with differentials coming from the $(\mathcal{C}^!, R^!)$-module structure on M.

To discuss Hochschild cohomology, however, we must discuss the "relative" case, i.e. find the Koszul dual of the category of $\mathcal{C}$-algebras over R. In this case, the derived category of abelian objects is the category of A_1-modules. This is because abelian objects are of the form

$$(55) \qquad\qquad\qquad\qquad R \oplus M$$

where M is a $(\mathcal{C}, R)$-module; the $\mathcal{C}$-algebra structure on (55) is defined by multiplying sn n-tuple with all elements in R via the $\mathcal{C}$-algebra structure of R, n-tuples with one elements in M and others in R by the $(\mathcal{C}, R)$-module structure on M, and setting the products of n-tuples with more than one entry in M to 0. Moreover, a map from a $\mathcal{C}$-algebra Q under R to (55) is a derivation $Q \to M$ in the obvious sense.

Now there is a notion of $(\mathcal{C}^!)^\vee$-coalgebras in the category of A_1-modules, where A_1 is as above (i.e. the fibrant replacement of the operad fibered over $\mathcal{C}$ associated with R - work internally in the category of chain complexes for the moment). The structure maps of such objects X include maps from X to iterated tensor products of X (indexed over elements of $\mathcal{C}$) in the category of A_1-modules (for a single such product, see (41)). The derived category of $(C^!)^\vee$-coalgebras over $A(0)$ in the category of A_1-modules is the derived Koszul transform of the category of $\mathcal{C}$-algebras.

There is a small remark to be made about base points: there is an equivalence of categories between augmented algebras and non-unital algebras. In the above description of based Koszul duality we used the augmented approach to describe the Koszul transform (or dual), whereas in this paragraph we used the non-unital approach.

To describe explicitly the Koszul dual of an $\mathcal{C}$-algebra under R, we first discuss the case of a free $\mathcal{C}$-algebra CX under R, i.e. with a map $CX \to R$. Then the target of the universal derivation from CX into a $(\mathcal{C}, R)$-module is the free $(\mathcal{C}, R)$-module on X. Denoting such free module by $F_R X$ for now, we see that the Koszul dual of

a $\mathcal{C}$-algebra Q under R is

(56) $$C^R_! Q := B(F_R, C, Q).$$

Note, however, that here the monad C is considered in the category of chain complexes under R, since the contraction $F_R C \to F_R$ depends on the map into R (although the module itself does not). Another way to rewrite (56) is as

$$A_1 \otimes C_! Q$$

with additional differentials expressing the structure of $\mathcal{C}$-algebra under R on Q.

We encourage the reader to consider the case of $\mathcal{C} = \mathcal{C}_1$: the Koszul transform of the category of associative algebras under R is the category of coalgebras under R in the category of R-bimodules, with respect to the internal product $\otimes_R$ of R-bimodules (R-bimodules are A_1-modules in this case). The Koszul dual of Q over R comes to

$$RHom_{Q \otimes Q^{Op}}(Ker(Q \otimes Q^{Op} \to Q), R \otimes R^{Op}).$$

In particular, for $Q = R$, we get just the kernel of the canonical map [16]

(57) $$R \otimes R^{Op} \to R.$$

In the case of $\mathcal{C} = \mathcal{C}_k$, the little k-cube operad, Kontsevich [9] calls for comparing Hochschild cohomology of a $\mathcal{C}$-algebra R to its Quillen cohomology in the category of $\mathcal{C}$-algebras under R, with coeficients in R. These cohomologies are obtained by applying $RHom_{A_1}(?, R)$ to R and $B(F_R, C, R)$, respectively. In particular, he suggests that the Hochschild cohomology complex can be obtained as a quotient of the Quillen cohomology complex by a certain ideal. Thus, we need a map of A_1-modules in the derived category

(58) $$A(0) \to \Sigma^k A(1).$$

By the 2-sided monad bar construction trick, it suffices to give a sufficiently natural such map in the case when R is a free $\mathcal{C}_k$-algebra $C_k(X)$. Replacing the associated operad over $\mathcal{C}_k$ of the $\mathcal{C}_k$-algebra R by a fibrant operad A over $\mathcal{C}_k$, we shall describe the Koszul transform of the map (58) with respect to Koszul transformation between A_1-modules and BA_1-comodules. In effect, the Koszul transform $M_!$ of an A_1-module is

$$B(*, A_1, M).$$

Now the recognition principle of [14] can be interpreted to say that

$$A(0)_! = B(*, A_1, A(0)) \simeq \Sigma^k(X_+).$$

Since obviously

$$(\Sigma^k A(1))_! \simeq S^k,$$

the Koszul transform of (58) is just the collapse map

$$\Sigma^k X_+ \to S^k.$$

We shall make one more remark, regarding the algebra A_1: there is in general no reason why this algebra should be (quasi)-isomorphic to its opposite. Because of that, there seems to be no reason why $A(0)$ should be a *right* A_1-module for a general (fibrant) operad A over $\mathcal{C}$, and hence there seems to be no natural notion of Hochschild *homology* in this generality (although, from a different point of view, this may be remedied by considering the framed little disks operad).

For $\mathcal{C} = \mathcal{C}_k$, one has $A(1) \cong A(1)^{Op}$ if the k-sphere is parallelizable. It would be interesting to see if Adams' Hopf 1 invariant theorem could be proven in this context. Clearly, this is a question in characteristic 2, and one could speculate that this could be related to expressing secondary Steenrod operations in terms of the little cube operads.

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