# ON THE DEFINITION AND K-THEORY REALIZATION OF A MODULAR FUNCTOR

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Abstract. We present a definition of a (super)-modular functor which includes certain interesting cases that previous definitions do not allow. We also introduce a notion of topological twisting of a modular functor, and construct formally a realization by a 2-dimensional topological field theory valued in twisted K-modules. We discuss, among other things, the $N = 1$-supersymmetric minimal models from the point of view of this formalism.

## 1. Introduction

This paper is being written on the 10th anniversary of the publication of the first author's paper [32], and it grew out of a project of writing a sequel of [32], solving certain questions posed there, and also commenting on how the material relates to some topics of current interest. During the process of writing the paper, the authors received many comments asking for examples of their theory. Supplemented with those examples, which required venturing into other fields of mathematics, the scope of the paper now exceeds by far what was originally intended.

The original aim of [32] was to work out the modularity behavior of certain series known in mathematical physics as partition functions of chiral conformal field theories with 1-dimensional conformal anomaly. The author intended to use the outline of Segal [50], and work things out in more mathematical detail. However, unexpected complications arose. In correspondence with P. Deligne [11] (see also [10]), the author learned that an important example of 1-dimensional conformal anomaly, known as the *Quillen determinant* of a Riemann surface, only satisfies the desired gluing axioms when considered as a *super-line* instead of just an ordinary complex line. A super-line is the same thing as a line, but with an additional bit of information, labelling it

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as either even or odd. In the coherence isomorphism $L \otimes M \cong M \otimes L$, if both $L$ and $M$ are odd, a sign of $-1$ is inserted. The fact that the determinant is a super-line and not a line has a subtle and profound implication on its partition function: the partition function is, in fact, 0. This phenomenon was well familiar in mathematical physics, but was somewhat subtle to capture rigorously.

The situation is even more convoluted in the case of another conformal field theory, known as the chiral fermion theory of central charge $c = 1/2$ on Riemann surfaces with spin structure. While the conformal anomaly is still "1-dimensional" (meaning "invertible"), it cannot be given consistent signs even when we use super-lines ([11]). The odd spinors turn out to require the use of the non-trivial element of the super-Brauer group $sBr(\mathbb{C}) \cong \mathbb{Z}/2$, and a consistent theory can be built using the 2-category $\mathcal{D}_0$ of Clifford algebras over $\mathbb{C}$, graded Morita equivalences and degree 0 isomorphisms.

In [32], a rigorous concept 1-dimensional (or invertible) modular functor was introduced which included the situations mentioned above. In fact, such structures were classified. An important question left open was how to generalize the concept beyond the 1-dimensional case. Segal [50] previously outlined a definition of a modular functor (and coined the term), but did not discuss the super- or Clifford cases, or, in fact, the coherence diagrams required. Examples from mathematical physics are, at least conjecturally, abundant (just to begin, see [5, 47, 45, 12]). Moreover, super- examples are also abundant, because many of the examples have supersymmetry (such as super-minimal models [22] or the super-WZW model [38]), and supersymmetry requires super-modular functors. (In this paper, we discuss modular functors which are super-, but not the stronger condition of super-symmetry, since that requires a rigorous theory of super-Riemann surfaces, which still has not been worked out in mathematical detail, cf. [9].) While a number of approaches to rigorous definitions of modular functors and similar structures were proposed [3, 49, 27, 25, 19], the question of the correct definition of a super-modular functor, and a Clifford modular functor, remained open. It is solved in the present paper.

There were several recent developments which raised interest in this topic, and related it to a field of homotopy theory, namely K-theory. In [24], D.Kriz, I.Kriz and P.Hu were considering a generalized (co)-homology version of Khovanov homology, an invariant in knot theory which is a "categorifiacation" of the Jones polynomial. Using topological field theory methods, [24] constructs a stable homotopy version of Khovanov homology; the construction is an alternative to a previous construction of Lipshitz and Sarkar [35], which used Morse theory. Invariance under the Reidemeister moves used for proving knot invariance only requires an *embedded topological field theory*, which is what makes a stable homotopy type realization possible. In [24], a more canonical K-theoretical version is also discussed, which uses a more complete topological quantum field theory; this version is related to modular functors.

From the point of view of [24], a topological modular functor is, roughly speaking, a 2-dimensional topological quantum field theory valued in 2-vector spaces. The *K-theory spectrum* $K$ is an object of stable homotopy theory analogous to a commutative ring (called an $E_\infty$ ring spectrum). For an $E_\infty$ ring spectrum, there is a concept of a module. The K-theory realization of [24] constructs, from a topological modular functor, a 2-dimensional topological quantum field theory valued in $K$-modules. The passage to stable homotopy theory requires a sophisticated device called *multiplicative infinite loop space machine*. While many versions of such a machine are known to algebraic topologists, a version which seemed flexible enough for discussing topological field theory was discovered only in 2006 by Elmendorf and Mandell [17]; it used the concept of a *multicategory*. Because of this, a definition of topological modular functors given in [24] uses multicategories.

In the present paper, we give a generalization of the definition from [24] to super- and Clifford modular functors, and construct their *$K$-module realization* extending the construction of [24]. Mathematical physics, in fact, makes a K-theoretical realization of modular functors desirable also because of the fact that modular functors also classify "Cardy branes" [7], which correspond to certain "boundary sectors" of conformal field theories. Witten [52] argued that branes should be classified by $K$-theory, which, in mathematics, therefore includes a K-theory realization of modular functors.

There was still another clue directly related to the super- and Clifford case: it was noticed [1, 20] that the double bar construction (i.e. on both 2-morphisms and 1-morphisms) of the Deligne 2-category $\mathcal{D}_0$ discussed above gives the classifying space of geometrical twistings of the $E_\infty$ ring spectrum $K$. In this paper, we construct this twisting space *as an $E_\infty$-space*. This observation is, in fact, essentially equivalent to constructing K-theory realizations of the invertible super- and Clifford modular functors discussed in [32]. But it also points to the need of a model of the $E_\infty$ ring spectrum $K$ which would handle all the elements of the "Picard group" $Pic(K) = sBr(\mathbb{C})$. Specifically, by Bott periodicity, $K$-theory is invariant under dimensional shift (suspension) by 2. The non-trivial element of $sBr(\mathbb{C})$ should be realized by a single suspension of $K$, and the second "tensor power" of this element should give $K$ again: this phenomenon does not arise, for example, in algebraic K-theory (where Bott periodicity involves a Tate twist), and hence cannot be captured by any "finite", or algebraic, construction. Fortunately, a model of K-theory which handles this case was discovered by Atiyah and Singer [2], (it was also used in [1, 20]). Making the Atiyah-Singer model work in the context of $E_\infty$ ring spectra requires some additional work, which is also treated in the present paper.

The paper [20] identifies the Verlinde algebra of the WZW model as the equivariant twisted K-theory of a (say, simply connected) compact Lie group $G$ acting on itself by conjugation. It therefore begs the question whether the fixed point spectrum of the twisted K-theory spectrum $K_{G,\tau}(G)$ itself is the K-theory realization of the modular functor of the WZW model. We address a slightly weaker question here: we show that the twisted $K$-theory realization depends only on a weaker structure of a *projective* modular functor. Using the machinery of modular tensor categories (as described, say, in [3]), we are able to construct the twisted $K$-theory data from the WZW model, although a more direct connection would still be desirable.

In particular, it is important to note tha the twisting comes from the fact that the modular functors of the WZW models are not topological, but holomorphic. The violation of topological invariance in a holomorphic modular functor is expressed by a single numerical invariant called *central charge*. In this paper, we extract the topological information in the central charge in an invariant we call *topological twisting*. It turns out to be a *torsion* invariant (it vanishes when the central charge is divisible by 4). We show that holomorphic modular functors can be realized as 2-dimensional quantum field theories valued in twisted K-modules, which is precisely the type of structure present in the Freed-Hopkins-Teleman spectrum $(K_{G,\tau}(G))^G$. This is interesting because of speculations about possible connections of 1-dimensional modular functors with dimensions of elliptic cohomology [4, 51]: in connection with those ideas, a torsion topological invariant associated with central charge was previously conjectured.

In view of these observations, we felt we should construct non-trivial examples at least of *projective* modular functors which would utilize the full Clifford generality we introduce. As a somewhat typical case, we discuss the example of the $N = 1$-supersymmetric minimal models. Getting projective Clifford modular functors from these examples rigorously is a lot of work. First of all, we need to identify the additional data on modular tensor categories which allow us to produce a Clifford modular functor. We treat this in Section 7, where we introduce the concept of a *Clifford modular tensor category*, and show how to use it to construct a projective Clifford modular functor. This construction is general. It also applies to other examples which will be discussed elsewhere. Then, in Section 8, we gather the relevant facts from the vertex algebra literature and verify that they fit the formalism. In the end, we do associate a projective Clifford modular functor, and hence a twisted $K$-theory realization, with the $N =$-supersymmetric minimal model super-vertex operator algebras.

The present paper is organized as follows: Multicategories are discussed in Section 2. A definition of a classical (not super- or Clifford) modular functor in the sense of [50] with multicategories is given in Section 3. The super- and Clifford version of a modular functor, which is the main definition of the present paper, is given in Section 4. The K-theory realization of topological modular functors is presented in Section 5. The K-theory realization of a general modular functor, with a discussion of central charge, is given in Section 6. Section 7 contains the concept of a Clifford modular tensor category, which is a refinement of a modular tensor category which produces a projective Clifford modular functor. In Section 8, we discuss how this applies to the case of super vertex algebras, and we discuss concretely the example of the $N = 1$-supersymmetric minimal models. Section 9 is an Appendix, which contains some techincal results we needed for technical reasons to construct realization. Notably, this includes May-Thomason rectification, and a topological version of the Joyal-Street construction. We also discuss singular vectors in Verma modules over the $N = 1$ NS algebra.

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## 2. Multicategories

In this paper, we study among other things the interface between topological field theories and stable homotopy theory. This requires what is known as *multiplicative infinite loop space theory*, which is a difficult subject to treat rigorously. As in [24], we use the approach of Elmendorf and Mandell, based on *multicategories*.

A multicategory [17] is the same thing as a multisorted operad. This means we are given a set $S$ and for each $n = 0, 1, 2, \dots$ a set $\mathcal{C}(n)$ with a map to $S^{\times n} \times S$ with a $\Sigma_n$-action fibered over the $\Sigma_n$-action on $S^n$ by permutation of factors, a composition

$$\mathcal{C}(n) \times_{S^n} (\mathcal{C}(k_1) \times \cdots \times \mathcal{C}(k_n) \to \mathcal{C}(k_1 + \cdots + k_n)$$

over the identity on $S^{k_1+\cdots+k_n} \times S$ and a unit

$$S \to \mathcal{C}(1)$$

over the diagonal map $S \to S \times S$. These operations are subject to the same commutative diagram as required in the definition of an operad [36].

In a multicategory, we refer to $S$ as the set of *objects* and to elements of $\mathcal{C}(n)$ over $(s_1, \dots, s_n, s)$, $s, s_i \in S$, as *multimorphisms*

$$(s_1, \dots, s_n) \to s. \tag{1}$$

The set of multimorphisms (1) will be denoted by $\mathcal{C}(s_1, \dots, s_n; s)$. A *multifunctor* from a multicategory $\mathcal{C}$ with object set $S$ and $\mathcal{D}$ with object set $T$ consists of a map $F : S \to T$ and maps $\mathcal{C}(n) \to \mathcal{D}(n)$ over products of copies of $F$, which preserves the composition.

We will commonly use multicategories *enriched* in a symmetric monoidal category $Q$. This is a variation of the notion of a multicategory where the set of objects $S$ remains a set, but the multimorphisms $(s_1, \dots, s_n) \to s$ form objects $\mathcal{C}(s_1, \dots, s_n, s)$ of the category $Q$. Multicomposition then takes the form

$$\mathcal{C}(\dots) \otimes \mathcal{C}(\dots) \otimes \dots \mathcal{C}(\dots) \to \mathcal{C}(\dots)$$

where $\otimes$ is the symmetric monoidal structure in $Q$. The unit is a morphism

$$1 \to \mathcal{C}(s, s)$$

where 1 is the unit of $\otimes$. The required diagrams are still the obvious modifications of diagrams expressing operad axioms.

In particular, an ordinary multicategory is a multicategory enriched in the category of finite sets, with the symmetric monoidal structure given by the Cartesian product. Enrichments over the categories of topological spaces or simplicial sets are so common they are almost not worth mentioning. Multicategories enriched in the category of categories (or groupoids) and functors (with the Cartesian product as symmetric monoidal structure) also occur in Elmendorf-Mandell's approach to multiplicative infinite loop space theory. These notions however admit *weak versions*, which are trickier. A weak multicategory (or multifunctor) is a modification of the respective notion enriched in categories (or groupoids) where the respective axiom diagrams are required only to commute up to natural isomorphisms (called *coherence isomorphisms*). These isomorphisms must satisfy certain *coherence diagrams*, the precise definition of which is technical, and will be relegated to the Appendix (Section 9), along with a construction called *May-Thomason rectification*, which allows us to replace them by the corresponding strict notions in an appropriate sense. In Section 6, we will also briefly need to discuss weak versions of multicategories and multifunctors enriched in (strict) 2-categories. To complement weak multicategories, we shall call multicategories strictly enriched in groupoids *strict*. Strict multicategories can be converted to multicategories enriched in topological spaces by taking the nerve (bar construction) on 2-morphisms. We shall denote this operation by $B_2$.

It may be good, at this point, to note that weak multifunctors form a weak 2-category: A *1-morphism* of two weak multifunctors $\Phi$, $\Psi$ consists of the following data: for objects $x$, a 1-morphism $F : \Phi(x) \to \Psi(x)$ and for a 1-multimorphism $M : (x_1, \dots x_n) \to y$, 2-isomorphisms

$$\phi : M \circ (F, \dots, F) \cong F \circ M.$$

There are "prism-shaped" coherence diagrams required to be formed by these 2-isomorphisms $\phi$ and the coherence diagrams of multifunctors.

A 2-*isomorphism* of 1-morphisms $F$, $G$ of multifunctors consists of the following data: for every object $x$, a 2-isomorphism $F(x) \cong G(x)$ which commute with the coherence isomorphisms $\phi$ of the 1-morphisms $F$, $G$.

Recall further that a *weak isomorphism* (or *equivalence*) between two objects $x, y$ of a weak 2-category is a pair of 1-morphisms $x \to y$, $y \to x$ whose compositions are 2-isomorphic to the identities.

We will also use the notion of $\star$-*categories*. A $\star$-category is a multicategory in which for every $s_1, \dots, s_n \in S$, there exists a *universal* multimorphism $\iota : (s_1, \dots, s_n) \to s_1 \star \dots \star s_n$, i.e. for every multimorphism $\phi : (s_1, \dots\ s_n) \to t$, there exists a unique morphism $\psi : (s_1 \star \dots \star s_n) \to t$ such that $\psi \circ \iota = \phi$ (here we write $\circ$ for the composition in the obvious sense. The case of $n = 0$ is included, we denoted the empty $\star$-product by $\mathbf{1}$. A $\star$-*functor* is a weak multifunctor which preserves the $\star$-product. (Note that since the $\star$-product is defined by universality, there is no need to discuss coherences here.)

There is also a corresponding weak version in which $\psi \circ \iota \cong \phi$, and after a choice of that natural 2-isomorphism, $\iota$ is determined up to unique 2-isomorphism making the resulting 2-diagram commute. The definition of a $\star$-functor remains unchanged.

A symmetric monoidal category with symmetric monoidal structure $\otimes$ determines a $\star$-category by letting the multimorphisms $(a_1, \ldots, a_n) \to a$ be the morphisms $a_1 \otimes \cdots \otimes a_n \to a$. The notion of a $\star$-category, however, is more general. Let $a_{11}, \ldots a_{nk_n}$ be objects of a $\star$-category. Then we have multimorphisms $(a_{i1}, \ldots, a_{ik_i}) \to a_{i1} \star \cdots \star a_{ik_i}$. By the composition property, we then have a multimorphism

$$(a_{11}, \ldots, a_{nk_n}) \to (a_{11} \star \cdots \star a_{1k_1}) \star \cdots \star (a_{n1} \star \cdots \star a_{nk_n}).$$

By universality, we get morphisms

$$a_{11} \star \cdots \star a_{nk_n} \to (a_{11} \star \cdots \star a_{1k_1}) \star \cdots \star (a_{n1} \star \cdots \star a_{nk_n}) \tag{2}$$

and the category is symmetric monoidal when the morphisms (2) are all isomorphisms (including the case when some of the $k_i$'s are equal to 0).

By a *weak symmetric monoidal category* we shall mean a weak $\star$-category in which the 1-morphisms (2) are *equivalences*, which means that there exists an inverse 1-morphism with both compositions 2-isomorphic to the identity.

The main purpose of using multicategories in our context comes from the work of Elmendorf and Mandell [17] who constructed a strict multicategory $Perm$ of permutative categories, where multimorphisms are, roughly, multilinear morphisms (permutative categories are a version of symmetric monoidal categories where the operation is strictly associative, see [37]; the Joyal-Street construction [30], which we will briefly discuss below in Section 9, allows us to rectify symmetric monoidal categories into permutative categories).

Elmendorf and Mandell further discuss a *realization multifunctor*

$$\mathcal{K} : B_2(Perm) \to \mathcal{S} \tag{3}$$

where $\mathcal{S}$ is the topological symmetric monoidal category of symmetric spectra ([17]). The multifunctor $\mathcal{K}$ is unfortunately not a $\star$-functor. In fact, more precisely, the category $Perm$ is a $\star$-category, but only in the weak sense. In effect, the weak $\star$-product of $n$ permutative categories $C_1, \ldots, C_n$ has as objects formal sums

$$(a_{11} \otimes \cdots \otimes a_{1n}) \oplus \cdots \oplus (a_{k1} \otimes \cdots \otimes a_{kn})$$

where $a_{ij} \neq 0$, and is freely generated by "tensor products" of morphisms in the categories $C_i$ and the required coherences, modulo the coherence diagrams prescribed for multimorphisms in $Perm$. On the other hand, a strong $\star$-product would require that all objects be of the form $a_1 \otimes \dots a_n$ (because those are the only objects on which the value of the universal 1-morphism is prescribed exactly); this is clearly impossible except in special cases.

However, the category $\mathcal{S}$ has a Quillen model structure [15], and in particular a notion of equivalence; the functor $\mathcal{K}$ is a *homotopy $\star$-functor* in the sense that the map

$$\mathcal{K}(a_1) \star \cdots \star \mathcal{K}(a_n) \to \mathcal{K}(a_1 \star \cdots \star a_n)$$

coming from the multimorphism

$$(\mathcal{K}(a_1), \dots, \mathcal{K}(a_n)) \to \mathcal{K}(a_1 \star \cdots \star a_n)$$

by the fact that $\mathcal{K}$ is a multifunctor is an equivalence.

## 3. Examples of multicategories. Naive modular functors

In this section, we shall discuss a number of examples of weak $\star$-categories, and will define a modular functor as a weak $\star$-functor between appropriate weak $\star$-categories (more precisely, a stack version will be needed to express holomorphic dependence, but we will discuss this when we get there).

The first kind of weak $\star$-categories which we will use are $1+1$-dimensional cobordism categories. Recall that if a boundary component $c$ of a Riemann surface $X$ is parametrized by a diffeomorphism $f : S^1 \to c$ then $c$ is called *outbound* (resp. *inbound*) depending on whether the tangent vector $\iota$ at 1 in the direction of $i$ is $i$ or $-i$ times a tangent vector of $X$ on the boundary pointing outside.

The weak $\star$-category $\mathcal{A}^{top}$ has objects finite sets, multimorphisms from $S_1, \dots, S_n$ to $T$ Riemann surfaces with real-analytically parametrized inbound boundary components labeled by $S_1 \amalg \cdots \amalg S_n$ and outbound boundary components labeled by $T$, and 2-morphisms are isotopy classes of diffeomorphisms preserving orientation and boundary parametrization. The operation $\star$ is, in fact, $\amalg$, and this makes $\mathcal{A}^{top}$ a weakly symmetric monoidal category.

A variant is the weak $\star$-category $\mathcal{A}$ which has the same with the same objects and multimorphisms as $\mathcal{A}^{top}$, but with 2-morphism holomorphic isomorphism preserving boundary component parametrization. In order for this weak multicategory to be a weak $\star$-category, we must consider a disjoint union of finitely many copies of $S^1$ a (degenerate) Riemann manifold where the copies of $S^1$ are considered both inbound and outbound boundary components (parametrized identically). Again, $\mathcal{A}$ is a weak symmetric monoidal category.

To model our concept which approaches most the original outline of Segal's concept of a modular [50], consider the weak $\star$-category $\mathcal{C}$ whose objects are finite sets, multimorphisms $(S_1, \dots, S_n) \to T$ are $T \times (S_1 \times \dots \times S_n)$-matrices of finite-dimensional $\mathbb{C}$-vector spaces, and 2-morphisms are matrices of isomorphisms of $\mathbb{C}$-vector spaces.

Then, a *naive topological modular functor* is a weak $\star$-functor

$$\mathcal{A}^{top} \to \mathcal{C}.$$

To eliminate the word "topological" means to replace $\mathcal{A}^{top}$ with $\mathcal{A}$, but then we want to include some discussion of holomorphic dependence on the Riemann surface. Then, there is a slight problem with the degenerate Riemann surfaces. We can, for example, consider the category of subsets of $\mathbb{C}^n$ (with $n$ varying) which are of the form $U \cup S$ where $U \subseteq \mathbb{C}^n$ is an open subset, and $S$ is a finite subset of the boundary of $U$ and continuous maps

$$U \cup S \to V \cup T$$

which are holomorphic on $U$. Consider the Grothendieck topology $\mathcal{G}$ on this category where covers are open covers. Then consider the stack $\widetilde{\mathcal{A}}$ where sections over $U \cup S$ are maps continuous maps $f$ from $U \cup S$ into the Teichmüller space of Riemann surfaces with parametrized boundary components (including the degenerate Riemann surfaces) which are holomorphic on $U$, and morphisms are continuous families of isomorphisms parametrized over $U \cup S$ which are holomorphic on $U$. Consider also the stack $\widetilde{\mathcal{C}}$ whose sections over $U \cup S$ are continuous vector bundles over $U \cup S$ with holomorphic structure on $U$, and continuous isomorphisms of vector bundles on $U \cup S$, which are holomorphic on $U$.

Again provisionally, then, a *naive modular functor* is a morphism of stacks

$$\widetilde{\Phi} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{C}}$$

the sections of which over any $U \cup S \in Obj(\mathcal{G})$ are weak multifunctors. We will continue to denote the sections over a point by

$$\Phi : \mathcal{A} \to \mathcal{C}.$$

The set $S = \Phi(*)$ is called the *set of labels*. One also usually includes the normalization condition that there be a special label $1 \in S$ where for the unit disk $D$ (with constant boundary parametrization), one $\Phi(D)(s)$ is 1-dimensional for $s = 1$ and trivial for $1 \neq s \in S$.

It was P.Deligne [10, 11] who first discovered that this definition of a modular functor is insufficiently general in the sense that it does not include the case of the Quillen determinant [48], which was meant to be one of the main examples discussed in [50]. For the Quillen determinant, the set of labels has a single element 1 and the value of the multifunctor on any 1-morphism is a 1-dimensional $\mathbb{C}$-vector space (called the *Quillen determinant line*), but Deligne observed that in order for the gluing to work (in our language, for the multifunctor axioms to be satisfied), the Quillen determinant line must be a *super-line*, i.e. must be given a $\mathbb{Z}/2$-grading where a permutation isomorphism switching the factors in the tensor product of two odd lines is $-1$. He further discovered that if this is allowed, the Quillen determinant line becomes non-canonical ([32]). We introduce the machinery necessary to capture that situation in the next section, but it turns out that the appropriate generality is even greater.

## 4. Clifford algebras and modular functors

Deligne also noticed that the situation is even worse with the invertible chiral fermion (of central charge $c = 1/2$ - see Section6 for a more detailed discussion of the central charge) on Riemann surfaces with spin structure ([32]). Although this modular functor is invertible under the tensor product, there is no consistent description in terms of lines or super-lines, and one must consider irreducible Clifford modules. This leads to the definitions we make in this section. With the most general definition, we will then construct the K-theory realization.

**Remark:** A somewhat confusing aspect of the chiral fermion is that there also exists a naive chiral fermion modular functor (of central charge $c = 1/2$) with has three labels [12] and therefore is not invertible. That example is of lesser significance to us, and will not be discussed further.

Let us recall that a *spin structure* on a Riemann surface $X$ (with boundary) is a square root of the tangent bundle $\tau$ of $X$, i.e. a complex holomorphic line bundle $L$ together with an isomorphism $L_X \otimes_{\mathbb{C}} L_X \cong \tau_X$. A *spin-structure* on a real 1-manifold $Y$ is a real line bundle $L_Y$ together with an isomorphism $L_Y \otimes_{\mathbb{R}} L_Y \cong \tau_Y$. It is important that a Riemann surface $X$ with spin structure and with boundary canonically induces a spin structure on the boundary $\partial X$: Let $L_{\partial X}$ consist of those vectors of $L_X|_{\partial X}$ whose square is $i$ times a tangent vector of $X$ perpendicular to the boundary and pointing outside.

Recall that $S^1$ has two spin structures called *periodic* and *anti-periodic*, depending on whether the bundle $L_{S^1}$ is trivial or a Möbius strip. The induced spin structure on the boundary of a disk is antiperiodic. When parametrizing a boundary component $c$ of a Riemann surface with spin structure (we will, again, restrict attention to real-analytic parametrizations), it is appropriate for our purposes to specify a *parametrization with spin*, i.e. a diffeomorphism $f : S^1 \to c$ where $S^1$ is given a spin structure, together with an isomorphism $L_{S^1} \to L_c$ over $f$, which squares to $Df$.

Now there are multicategories $\mathcal{A}_{spin}$ and $\mathcal{A}_{spin}^{top}$ whose objects are sets $S$ with a map to $\{A, P\}$, standing for "periodic" and "antiperiodic" (the inverse images of $A$, $P$ will be denoted by $S_A$, $S_P$). 1-multimorphisms $(S_1, \dots, S_n) \to T$ are Riemann surfaces with spin with parametrized boundary components with inbound resp. outbound boundary components indexed by $S_1 \amalg \dots \amalg S_n$ resp. $T$, with matching spin structures. 2-isomorphisms in $\mathcal{A}_{spin}$ are holomorphic isomorphisms $f$ with spin (i.e. with given square roots of $Df$) which is compatible with boundary parametrizations. 2-isomorphisms in $\mathcal{A}_{spin}^{top}$ are isotopy classes of diffeomorphisms with spin: for this purpose, it is more helpful to interpret spin structure equivalently as an $\widetilde{SL_2(\mathbb{R})}$-structure on the tangent bundle where $\widetilde{SL_2(\mathbb{R})}$ is the double cover of $SL_2(\mathbb{R})$. Then a diffeomorphism with spin is defined as a diffeomorphism over which we are given a map of the associated principal $\widetilde{SL_2(\mathbb{R})}$-bundles.

Again, in the case of $\mathcal{A}_{spin}$, $S^1_A$ and $S^1_P$, which are copies of $S^1$ with either spin structure, must be considered to be (degenerate) Riemann surfaces with spin structure, where both inbound and outound boundary components are the same, with identical parametrizations (including identical spin). Then we can form a stack $\widetilde{\mathcal{A}_{spin}}$ analogously to the stack $\widetilde{\mathcal{A}}$ in the previous section, over the same Grothendieck topology.

Based on ideas of P.Deligne [11, 32], to capture examples such as the invertible chiral fermion, we introduce the weak $\star$-category $\mathcal{D}$ whose objects are data of the form $(S, A_s)$ where for every $s \in S$, $A_s$ is a super-central simple algebra, 1-multi-morphisms $((S_1, A_s), \dots (S_n, A_s)) \to (T, B_t)$ are $T \times (S_1 \times \dots \times S_n)$-matrices, the $(t, (s_1, \dots, s_n))$ entry being a $(B_t, A_{s_1} \otimes \dots \otimes A_{s_n})$-bimodule, and 2-isomorphisms are graded isomorphisms of bimodules.

Here, super-central simple algebras over $\mathbb{C}$ can be defined intrinsically, but for our purposes we may define them as *Clifford algebras*, i.e. $\mathbb{Z}/2-$ *graded* algebras graded-isomorphic to $\mathbb{C}$-algebras of the form $C_n = \mathbb{C}[x_1, \dots, x_n]/(x_j^2 - 1, x_j x_k + x_k x_j, j \neq k)$, where the degrees of the generators $x_j$ are odd. A Clifford algebra is *even* (resp. *odd*) depending on whether $n$ is even or odd. It is important to recall the *graded tensor product* of algebras or (bi)modules. This is the ordinary tensor product, but the interchange map $T : M \otimes N \to N \otimes M$ is defined by

$$T(x \otimes y) = (-1)^{deg(x)deg(y)} y \otimes x \tag{4}$$

on homogeneous elements $x \in M$, $y \in N$. This definition is applied in defining the $\Sigma_n$-action on 1-multi-morphisms, and also when making, for an $A$-module $M$ and $B$-module $N$, $M \otimes N$ an $A \otimes B$-module. The same applies to bimodules, and is used in defining the composition of 1-multimorphisms in $\mathcal{D}$.

Similarly as in the last section, we have a stack $\widetilde{D}$ over the Grothendieck topology $\mathcal{G}$ where sections over $U \cup S$ are matrices of holomorphic bundles of $(B_t, A_{s_1} \otimes \cdots \otimes A_{s_n})$-bimodules (for some $(S_1, A_s), \dots (S_n, A_s), (T, B_t)$), which are interpreted as holomorphic principal bundles with structure group

$$\prod GL_{m(s_1,\dots,s_n,t)}(B_t \otimes (A_{s_1} \otimes \cdots \otimes A_{s_n})^{Op})$$

where $m(s_1, \dots, s_n, t)$ are fixed non-negative integers.

We now define a *topological modular functor* as a $\star$-functor

$$\mathcal{A}_{spin}^{top} \to \mathcal{D}.$$

Analogously to the last section, a *modular functor* is a morphism of stacks

$$\widetilde{\Phi} : \widetilde{\mathcal{A}_{spin}} \to \widetilde{\mathcal{D}}$$

the section of which over an object of $\mathcal{G}$ form a $\star$-functor. The sections over a point will still be denoted by

$$\Phi : \mathcal{A}_{spin} \to \mathcal{D}.$$

The data $(S_A, A_s) = \Phi(* \mapsto A)$, $(S_P, A_s) = \Phi(* \mapsto P)$ are again referred to as sets of antiperiodic and periodic labels (decorated with Clifford algebras). Again, one may include a normalization condition that there exists exactly one label $(1, \mathbb{C})$ on which the module $\Phi(D)$ where $D$ is the outbound unit disk 1-multimorphism is 1-dimensional even, while on the other labels it is 0.

**Example:** The *chiral fermion* of central charge $c = 1/2$ which was considered in [32] is an example of a modular functor in the sense just defined. This is a theorem of [32]. In fact, this modular functor is *invertible* in the following sense:

There is an operation of a tensor product of modular functors; we call a modular functor $\Phi$ *invertible* if there exists a modular functor $\Psi$ such that $\Phi \otimes \Psi \sim 1$ where 1 is the modular functor with one antiperiodic and one periodic label, and all applicable 1-multimorphisms going to

$\mathbb{C}$. Here $\sim$ denotes a weak isomorphism of multifunctors covered by an equivalence of stacks.

Invertible modular functors $(\widetilde{\Phi}, \Phi)$ can be characterized as those for which $\Phi$ factors as

$$\Phi_0 : \mathcal{A}_{spin} \to \mathcal{D}_0 \subset \mathcal{D},$$

where $\mathcal{D}_0$ is the sub- weak multicategory of $\mathcal{D}$ whose objects are of the form $* \mapsto A$ for a Clifford algebra $A$, 1-multimorphisms are ($1\times$-matrices of) Morita equivalences, and 2-isomorphism are isomorphisms of bimodules. Recall that a *Morita equivalence* is a graded $A, B$-bimodule $M$ such that $M \otimes_B ?$ is an equivalence of categories between finitely generated $B$-modules and finitely generated $A$-modules.

The weak multi-category $\mathcal{D}_0$ will play a role in the next section, in connection with twistings of K-theory. In fact, invertible modular functors were completely classified in [32]. In particular, it was proved there that all invertible modular functors are weakly isomorphic to tensor products of tensor powers of the chiral fermion, and topological modular functors.

**Remark:** At least conjecturally, there should be a large number of examples which use the full generality of modular functors as defined here. For example, supersymmetric modular functors, such as $N = 1$ and $N = 2$-supersymmetric minimal models [22], which are irreducible representations of of certain super-algebras containing the Virasoro algebra, are almost certainly modular functors in our sense (and because of the super-symmetry, require the full scope of our formalism).

In this paper, we will discuss the case of the $N = 1$ supersymmetric minimal models, and will show that they give rise to at least a projective Clifford modular functors.

We do not discuss supersymmetry in this paper. One reason is that it requires some work on super-moduli spaces of super-Riemann surfaces, which still has not been done rigorously; the best reference available is the outline due to Crane and Rabin [9].

## 5. The Atiyah-Singer category and K-theory realization

In this section, we will describe how one can extract K-theory information out of a modular functor. The strategy is to construct a suitable weak $\star$-functor

$$\mathcal{D} \to Perm, \tag{5}$$

which, composed with the Elmendorf-Mandell multifunctor (3), would produce a homotopy $\star$-functor

$$B_2\mathcal{D} \to \mathcal{S},$$

which could be composed with a topological modular functor to produce a functor

$$B_2\mathcal{A}^{top}_{spin} \to \mathcal{S}. \tag{6}$$

(In case of modular functors, the superscript *top* would be dropped.)

There is, in fact, an obvious construction in the case of naive modular functors: We may define a weak $\star$-functor

$$\mathcal{C} \to Perm \tag{7}$$

on objects by

$$S \mapsto \prod_S \mathbb{C}_2$$

where $\mathbb{C}_2$ is the category of finite-dimensional $\mathbb{C}$-vector spaces and isomorphisms (topologized by the analytic topology on morphisms). A 1-multimorphism is then sent to the functor given by "matrix multiplication" (with respect to the operations $\oplus$ and $\otimes$) by the given matrix of finite-dimensional vector spaces. It is clear how matrices of 2-isomorphisms correspond to natural isomorphisms of functors.

As mentioned above, composing with (3), we get a multifunctor

$$B_2\mathcal{C} \to \mathcal{S}, \tag{8}$$

but using the following trick of Elmendorf and Mandell, we can in fact improve this, getting a functor into $K$*-modules* where $K$ denotes the $E_\infty$ ring spectrum of periodic K-theory (in fact, in the present setting, if we dropped the topology on morphisms of $\mathbb{C}_2$, it could just as well be the $E_\infty$ ring spectrum of algebraic K-theory of $\mathbb{C}$, which enjoys an $E_\infty$ map into $K$).

Let $Q$ be any multicategory. Consider a multicategory $\overline{Q}$ which has the objects of $Q$ and one additional object $*$. There is one multimorphism

$$(\underbrace{*, \dots, *}_{n}) \to * \tag{9}$$

for each $n$ and for every multimorphism

$$(a_1, \dots, a_n) \to b$$

in $Q$, a single multimorphism

$$(*, \dots, *, a_1, *, \dots, *, \dots a_n, *, \dots, *) \to b \tag{10}$$

for every fixed numbers of $*$'s inserted between the $a_i$'s. Composition is obvious. Similar constructions obviously also apply to enriched and weak multicategories.

Now if we have a weak multifunctor $\overline{Q} \to Perm$, then its restriction to the category with a single objects $*$ and multimorphisms (9) realizes to an $E_\infty$ ring spectrum $R$, and the restriction weak multifunctor $Q \to Perm$, which realizes to a multifunctor

$$B_2Q \to \mathcal{S},$$

is promoted to a multifunctor

$$B_2Q \to R - modules$$

(using the "strictification" Theorem 1.4 of [17]).

In the case of (7), we may define a weak multifunctor

$$\overline{\mathcal{C}} \to Perm \tag{11}$$

simply by sending $*$ to $\mathbb{C}_2$. The 1-morphisms (9), (10) are sent simply to tensors with the vector spaces corresponding to the $*$ copies. In this case, $R$ is connective K-theory $k$. Thus, we can promote (8) to a homotopy $\star$-functor to the multicategory of $k$-modules. By localizing with respect to the Bott element ([16]), we can further pass from $k$-modules to $K$-modules.

This suggests to construct the weak multifunctor (5) directly analogously to (7), i.e. to let

$$(S, A_s) \mapsto \prod_S A_s - Mod$$

where $A_s - Mod$ is the permutative category of finitely generated graded $A_s$-modules and graded isomorphisms of modules. Indeed, this does produce a weak multifunctor of the form (5), but *this is the wrong construction*; it does not, for example, restrict to (7) (note that $\mathcal{C}$ is a sub- weak multicategory of $\mathcal{D}$). In fact, permutative category of finitely generated graded $\mathbb{C}$-modules is equivalent to the product of *two copies* of $\mathbb{C}_2$.

There is a good heuristic argument why no *finite-dimensional* construction of a multifunctor (5) can possibly be what we want. It relates to the weakly symmetric monoidal category $\mathcal{D}_0$ which was discussed in the last section: The $E_\infty$ symmetric monoidal category $B_2\mathcal{D}_0$ has two objects $\{even, odd\}$ (with the expected product), and with automorphism groups of homotopy type $\mathbb{Z}/2 \times K(Z, 2)$; it can be interpreted as $B_2$ of the category of super-lines and isomorphisms (with the analytic topology). This suggests that the $E_\infty$ space $B(B_2\mathcal{D}_0)$ is a geometric model of the space of *twistings* of $K$-theory as considered in [20]. We will, in fact, be able to make that more precise below.

For now, however, let us look at the chiral fermion modular functor example. If we want to construct a realization of this into a weak $\star$-functor into $K$-modules, then the periodic label, which the modular functor sends to an odd Clifford algebra, should be twisted by a shift of dimension by 1, i.e. it should be weakly equivalent to the $K$-module $\Sigma K$. Therefore, the homotopy $\star$-functor

$$B_2\mathcal{D}_0 \to K - Mod$$

uses in substantial ways the relation

$$\Sigma K \wedge_K \Sigma K \sim K,$$

which is Bott 2-periodicity. This clearly indicates that the construction cannot have a direct algebraic K-theory analog, in which Bott periodicity is only valid with a Tate twist.

Therefore, one must bring to bear the full machinery of topological K-theory. The most convenient model for this purpose seems to be a minor modification of the construction of Atiyah and Singer [2]. Let $C$ be a Clifford algebra (over $\mathbb{C}$). By a *Hilbert $C$-module* we shall mean a $\mathbb{Z}/2$-graded complex (separable) Hilbert space $H = H_{even} \oplus H_{odd}$ together with a morphism of graded $\mathbb{Z}/2$-graded $C^*$-algebras $C \to B(H)$ where $B(H)$ is the $\mathbb{Z}/2$-graded $C^*$-algebra of bounded linear operators on $H$. (Recall that the canonical involution on $C$ sends $x \mapsto x$ for $x$ even and $x \mapsto -x$ for $x$ odd.)

Now we shall define a symmetric monoidal category $\mathcal{F}(C)$ in which both the sets of objects and morphisms are topologized; some basic facts about such categories, including the Joyal-Street construction (making them into permutative categories) will be discussed in the Appendix (Section 9).

The space $Obj(\mathcal{F}(C))$ is a disjoint union over (finite or infinite-dimensional) Hilbert $C$-modules $H$ of spaces $\mathcal{F}(H)$ defined as follows: When $C$ is even, $\mathcal{F}(H)$ consists of all homogeneous odd skew self-adjoint Fredholm operators $F : H \to H$ which anticommute with all odd elements of $C$. When $C$ is odd, $\mathcal{F}(H)$ consists of all homogeneous odd skew self-adjoint Fredholm operators $F : H \to H$ which anticommute with all odd elements of $C$ such that $iF$ is neither positive definite nor negative definite on any subspace of finite codimension. (Note that in the odd case, this in particular excludes the possibility of $H$ being finite-dimensional.) In both cases, the topology on $\mathcal{F}(H)$ is the induced topology from $B(H) \times K(H)$ via the map $F \mapsto (F, 1+F^2)$ where $B(H)$

is given the weak topology and $K(H)$ is the space of compact operators on $H$ with the norm topology. (At this point, we could equivalently just use the norm topology, but the more refined topology described above, which is due to Atiyah-Segal [1] is needed when considering the stack version of the multifunctor (5) which we are about to define.)

The space $Mor(\mathcal{F}(C))$ is a disjoint union over pairs $(H, K)$ of Hilbert $C$-modules of the spaces

$$\mathcal{F}(H) \times Iso(H, K)$$

where $Iso(H, K)$ is the space of metric isomorphisms of Hilbert $C$-modules with the norm topology. (Recall that when $H, K$ are infinite-dimensional, then $Iso(H, K)$ is contractible by Kuiper's theorem.)

Now the category $\mathcal{F}(C)$ is symmetric monoidal with the operation of direct sum $\oplus$. By a theorem of Atiayh and Singer [2], in fact, the spectrum associated with the symmetric monoidal category $\mathcal{F}(C)$ is $k$ when $C$ is even and $\Sigma k$ when $C$ is odd. (Thus, localizing at the Bott element produces the spectra we need.)

The weak $\star$-functor (5) can now be constructed as follows: On objects, we put

$$(S, A_s) \mapsto \prod_{s \in S} \mathcal{F}(A_s).$$

On 1-morphisms, we let a multimorphism $((S_1, A_s), \dots, (S_n, A_s)) \to (T, B_t)$ given by a matrix of bimodules $M_{t,(s_1,\dots,s_n)}$ send an $n$-tuple

$$((H_s, F_s | s \in S_1), \dots, (H_s, F_s | s \in S_n))$$

of pairs consisting of a Hilbert space and skew self-adjoint Fredholm operator to the $n$-tuple indexed by $t$ of Hilbert $A_t$-modules

$$\bigoplus_{s_i \in S_i} M_{t,(s_1,\dots,s_n)} \otimes_{A_{s_1} \otimes \dots \otimes A_{s_n}} H_{s_1} \widehat{\otimes} \dots \widehat{\otimes} H_{s_n}$$

(where $\widehat{\otimes}$ is the Hilbert tensor product) with the corresponding "product of skew self-adjoint Fredholm operators", as defined in [2]. This completes the definition of (5), and hence, in particular, of (6) for an arbitrary topological modular functor.

We may again use the Elmendorf-Mandell trick to extend the weak multifunctor (5) to $\overline{\mathcal{D}}$ by sending $*$ to $C_2$; combining with localization at the Bott element, we promote (6) to a weak $\star$-functor

$$B_2 \mathcal{A}^{top}_{spin} \to K - Mod,$$

which, for topological modular functors, is what we were asking for. For a general modular functor, we get, of course, the same thing with $\mathcal{A}^{top}_{spin}$ replaced by $\mathcal{A}_{spin}$, but that is somewhat unsatisfactory. We may, in fact, consider a Hilbert bundle version (following the lines of [20]) of the construction to obtain a sheaf version, but on the $K$-module side, we do not know how to preserve the holomorphic information, so the sections of 1-morphisms over a space $Y$ will be modules over $Map(Y, K)$. In addition to losing holomorphic information, this will only be a presheaf of spectra, satisfying the sheaf condition up to homotopy. Therefore, we clearly want to say something better for modular functors which are not topological. We will address that in the next section.

To conclude the present section, let us note that by restricting the functor (5) to $\overline{\mathcal{D}}$, and then restricting to $\overline{\mathcal{D}_0}$, we obtain a homotopy $\star$-functor

$$B_2(\mathcal{D}_0) \to K - modules.$$

This is one model of the "action" of the $E_\infty$ space $B(B_2\mathcal{D}_0)$ on the category of $K$-modules, as mentioned above.

## 6. Topological twisting and remarks on classification

In this section, we will address the question how to extract topological information from a (not necessarily topological) modular functor. Let $\widetilde{\mathcal{D}}_{proj}$ be the *projective version* of the stack $\widetilde{\mathcal{D}}$, i.e. sections over $U \cup S$ are principal bundles (holomorphic over $U$) with structure group equal to the product over the individual matrix entries of the (topological) automorphism groups of the respective Clifford modules, factored out by $\mathbb{C}^\times$, acting (on all the matrix entries simultaneously) by scalar multiplication. We see that the section of $\widetilde{\mathcal{D}}_{proj}$ over an object of $\mathcal{G}$ form a multicategory.

Note also that $\mathcal{A}^{top}_{spin}$ can also be promoted to a stack $\widetilde{\mathcal{A}^{top}_{spin}}$ which is simply equivalent to the quotient stack $[*/\Gamma]$ where $\Gamma$ is the appropriate mapping class group. We have then canonical projections of stacks

$$p : \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}_{proj}$$

and

$$q : \widetilde{\mathcal{A}_{spin}} \to \widetilde{\mathcal{A}^{top}_{spin}}.$$

The key observation is the following result due to G. Segal [50]:

**Lemma 1.** *Consider a modular functor $\Phi$ (with the normalization condition), there is a canonical morphism of stacks completing the following diagram:*

$$
\begin{tikzcd}
\widetilde{\mathcal{A}_{spin}} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "q"'] & \widetilde{\mathcal{D}} \arrow[d, "p"] \\
\widetilde{\mathcal{A}^{top}_{spin}} \arrow[r, dotted, "\widetilde{\Phi_{proj}}"'] & \widetilde{\mathcal{D}}_{proj}
\end{tikzcd}
\tag{12}
$$

*Proof sketch:* The idea is to cut out a small disk from a Riemann surface with boundary, and glue in an annulus (which can vary along a parametric set $U \cup S$). Then the modular functor takes on a non-zero value only for the unit label on the cut, and its value on the annulus (which must have the same labels on both boundary components) is also 1-dimensional. The gluing isomorphism then establishes a projective trivialization of the matrix of bundles given by $\widetilde{\Phi}$ on the given section of $\widetilde{A}_{spin}$ over $U \cup S$.

To prove consistency, we must show that the projective trivialization constructed does not depend on the choice of the holomorphic disk we cut out. To this end, consider a pair of pants with unit label on all boundary components. By the gluing isomorphism, again, the value of the modular functor on the pair of pants is 1-dimensional. This shows that the projective trivializations obtaining by gluing annuli on either of the boundary components of the pair of pants coincide.

$\square$

In diagram (12), since the source of the bottom row is topological, we can drop the $\widetilde{?}$, i.e. it suffices to consider the sections over a point

$$\Phi_{proj} : \mathcal{A}^{top}_{spin} \to \mathcal{D}_{proj}.$$

But how to realize this data topologically? While we could factor out $\mathbb{C}^\times$ from the morphisms of the categories $\mathcal{F}(C)$, (which is clearly related to twisted K-theory), those categories are no longer symmetric monoidal, so it is not clear how to apply the infinite loop space machine of Elmendorf and Mandell.

To remedy this situation, we shall, instead of factoring out the $\mathbb{C}^\times$ from the morphisms, add it as 3-morphisms to the target multicategory. More specifically, we consider a weak multicategory $\overline{\mathcal{D}}$ strictly enriched in groupoids, by which we mean a structure satisfying the axioms of a weak multicategory where the 2-morphisms between two

1-multimorphisms form a groupoid, and composition is functorial. The definition of $\overline{\mathcal{D}}$ is the same as the definition of $\mathcal{D}$ with an added space of 3-morphisms which is the Cartesian product of the space of 2-morphisms and $\mathbb{C}^{\times}$: An element $\lambda \in \mathbb{C}^{\times}$ acts on a 2-morphism by scalar multiplication.

We also introduce the concept of a *2-weak multifunctor*

$$Q \to W$$

where $Q$ is a weak multicategory and $W$ is a weak multicategory strictly enriched in groupoids in the above sense. This the weak version of the concept of a weak multifunctor, considered as a weak morphism of multisorted algebras of operad type with objects and 1-morphisms fixed, as considered below in Section 9. In other words, the 2-morphisms satisfy the axioms of a weak multifunctor where every equality of 2-morphisms prescribed by that structure is replaced by a 3-isomorphism. 3-isomorphisms are then required to satisfy coherence diagrams corresponding to situations where one operation on 2-morphisms in the concept of a weak multifunctor can be converted to another by a sequence of relations required by the structure in two different ways.

From this point of view, Lemma 1 gives a 2-weak multifunctor

$$\overline{\Phi} : \mathcal{A}^{top}_{spin} \to \overline{\mathcal{D}}. \tag{13}$$

Note that the data (13) are completely topological! We will refer to a 2-weak multifunctor (13) as a *projective modular functor*.

To construct a topological realization of (13), let $C$ be a Clifford algebra. We construct a category strictly enriched in groupoids $\overline{\mathcal{F}(C)}$ to have the same objects and 1-morphisms as $\mathcal{F}(C)$, and we let the space of 2-morphisms be the Cartesian product of the space of 1-morphisms with $\mathbb{C}^{\times}$; the 2-morphisms act, again, by scalar multiplication.

Then $\overline{\mathcal{F}(C)}$ is not a symmetric monoidal category enriched in groupoids: there is no way of adding two different 2-morphisms. Consider instead the strict 2-category $O^2\mathbb{C}^{\times}$ whose spaces of objects and 1-morphisms are $*$, and the space of 2-morphisms is $\mathbb{C}^{\times}$. Then there is an obvious "forgetful" strict 2-functor

$$U : \overline{\mathcal{F}(C)} \to O^2\mathbb{C}^{\times}, \tag{14}$$

which gives $\overline{\mathcal{C}}$ the structure of a symmetric monoidal category strictly enriched over groupoids over $O^2\mathbb{C}^{\times}$. Recall that a symmetric monoidal category $H$ over a category $K$ is a functor $H \to K$, a unit $K \to H$ and a product $\oplus : H \times_K H \to H$ which satisfy the usual axioms of a symmetric monoidal category. The version strictly enriched in groupoids is completely analogous. Further, the (topological) Joyal-Street construction allows us to rectify each symmetric monoidal category over a category into a permutative category over a category (which means that the symmetric monoidal structure is strictly associative unital), and similarly for the version strictly enriched in groupoids. Analogously to the theorem of Elmendorf and Mandell [17], we then have a strict multicategory

$$Perm/Cat$$

of permutative categories over categories, and the corresponding version strictly enriched in groupoids

$$(2-Perm)/(2-Cat).$$

Now we may construct from (13) a weak multifunctor strictly enriched in groupoids

$$\overline{\mathcal{D}} \to (2-Perm)/(2-Cat) \tag{15}$$

simply by the same construction as we used for (5), where on the level of 3-morphisms, we define composition by multiplication in $\mathbb{C}^{\times}$. Using $\overline{\Phi}$, we then obtain a 2-weak multifunctor

$$\mathcal{A}^{top}_{spin} \to (2-Perm)/(2-Cat). \tag{16}$$

Using the techniques described in Section 9, this can be rectified into a weak multifunctor strictly enriched in groupoids, so rectifying the 2-level and applying $B_3$, we get a weak multifunctor

$$\mathcal{A}^{top}_{spin} \to Perm/Cat. \tag{17}$$

Using the machine of Section 9 again, we can convert this to a strict multifunctor.

Now there is a relative version of the Elmendorf-Mandell machine, which produces a multifunctor

$$B_2 Perm/Cat \to \text{Parametrized symmetric spectra.} \tag{18}$$

The construction is on the formal level a fairly straightforward analog of the construction of Elmendorf-Mandell [17], although setting up a full model structure on symmetric parametrized spectra is actually quite tricky (see [6, 39]). We omit the details, as this would make the present paper disproportionately long.

Using also an analogue of the Elmendorf-Mandell module trick in the category of parametrized spectra, sending the label $*$ to the 2-permutative category $\overline{\mathbb{C}_2}$ over $O^2\mathbb{C}^\times$ given by finite-dimensional $\mathbb{C}$-vector spaces, isomorphisms and isomorphisms $\times\mathbb{C}^\times$, we can then obtain a multifunctor

$$B_2\mathcal{A}^{top}_{spin} \to \text{modules over twisted K-theory} \tag{19}$$

where by twisted K-theory we mean the parametrized $E_\infty$ ring spectrum $K/K(\mathbb{Z},3)$, constructed by applying the relative Elmendorf-Mandell machine to $\overline{\mathbb{C}_2}$, and localizing fiber-wise at the Bott element.

Note that because of the construction we used, we can actually say more about what happens to the twisting in (19). Let us compose (16) with the forgetful functor

$$(2-Perm)/(2-Cat) \to (2-Cat),$$

obtaining a 2-weak multifunctor

$$\mathcal{A}^{top}_{spin} \to (2-Cat). \tag{20}$$

Denote by $\mathcal{O}^2\mathbb{C}^\times$ the strong multicategory enriched in groupoids which has only one object and morphisms $O^2\mathbb{C}^\times$. Then we have an obvious forgetful strict multifunctor enriched in groupoids

$$\overline{\mathcal{D}} \to \mathcal{O}^2\mathbb{C}^\times. \tag{21}$$

Consider the 2-weak multifunctor

$$\Phi_{twist} : \mathcal{A}^{top}_{spin} \to \mathcal{O}^2\mathbb{C}^\times \tag{22}$$

given by the composition of (13) with (21). We will call the 2-weak multifunctor (22) the *topological twisting* associated with $\widetilde{\Phi}$. By definition, (20) is determined by (22): the objects go to $O^2\mathbb{C}^\times$, and the morphism go to the product, multiplied by another copy of $O^2\mathbb{C}^\times$ determined by (22) on morphisms. Thus, we obtain the following

**Theorem 2.** *A modular functor $\widetilde{\Phi}$ determines a multifunctor*

$$|\widetilde{\Phi}| : B_2\mathcal{A}^{top}_{spin} \to K/K(\mathbb{Z},3) - modules. \tag{23}$$

*Furthermore, the topological twisting determines a multifunctor $\phi$ from $B_2\mathcal{A}^{top}_{spin}$ to the multicategory with objects $*$ and morphisms $K(\mathbb{Z},3)$ (considered as an abelian group), with composition given by $K(\mathbb{Z},3)$-multiplication; on underlying spaces $K(\mathbb{Z},3)$, (23) on a space of multimorphisms is given by the product in the abelian group $K(\mathbb{Z},3)$, multiplied additionally by $\phi$.*

□

**Explanation:** Recall [39] that for a map of spaces $f : X \to Y$, there is a pullback functor parametrized spectra

$$f^* : Spectra/Y \to Spectra/X$$

which has a left adjoint denoted by $f_\sharp$ and a right adjoint denoted by $f_*$. (The situation with parametric modules is the same.) The map on multimorphisms given by Theorem 2 can be described as follows. Denote by $\mu = \mu_n : K(\mathbb{Z}, 3)^{\times n} \to K(\mathbb{Z}, 3)$ the multiplication, and let $\wedge$ denote the external smash-product of parametrized spectra (i.e. sending a parametrized spectrum over $X$ and a parametrized spectrum over $Y$ to a parametrized spectrum over $X \times Y$). We have the map

$$\phi : B_2\mathcal{A}^{top}_{spin}(S_1, \ldots, S_n; T) \to K(\mathbb{Z}, 3).$$

Let

$$\pi : B_2\mathcal{A}^{top}_{spin}(S_1, \ldots, S_n; T) \times K(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3)$$

be the projection, and let

$$j : B_2\mathcal{A}^{top}_{spin}(S_1, \ldots, S_n; T) \times K(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3)$$

be given by

$$(x, y) \mapsto \phi(x) \cdot y.$$

(Note that $B_2\mathcal{A}^{top}_{spin}(S_1, \ldots, S_n; T)$ is homotopically equivalent to the classifying space of a mapping class group.) Then the map on multimorphisms given by Theorem 2 is a map of $K/K(\mathbb{Z}, 3)$-modules of the form

$$j_\sharp\pi^*(\mu_\sharp(|\widetilde{\Phi}|(S_1) \wedge \cdots \wedge |\widetilde{\Phi}|(S_n)) \wedge_{\mu_\sharp(K/K(\mathbb{Z},3)\wedge\ldots\wedge K/K(\mathbb{Z},3))} K/K(\mathbb{Z}, 3))$$
$$\to |\widetilde{\Phi}|(T).$$

**Example:** At present, projective modular functors are easier to construct than modular functors. For example, modular tensor categories, as defined in [3], give rise to projective versions of the "naive" modular functors considered in Section 3. The authors of [3] give an exact statement of coherence diagrams of a modular functor as an exercise to the reader, and Section 3 of the present paper can be interpreted as one approach to a solution of that exercise. The proof of [3], giving a passage from a modular tensor category to a projective version of a modular functor in the sense of Section 3, in any case, applies.

From this point of view, we can make contact with the work of Freed-Hopkins-Teleman [20]. They compute the equivariant twisted $K$-theory groups $K^*_{G,\tau}(G)$ where $\tau$ is a "regular" twisting in an appropriate sense, and $G$ is a compact Lie group acting on itself by conjugation. At least for, say, compact Lie groups with torsion free fundamental group, this coincides with the Verlinde algebra obtained by taking dimensions of the vector spaces in the (naive - although there is also an $N = 1$-supersymmetric version) modular functor corresponding to the chiral WZW model.

The chiral WZW model is known to give rise a modular tensor category ([3, 26]), and hence gives rise to a projective modular functor. The Verlinde algebra in this case has been computed in the physics literature (see [12] for a survey and original references), and the known answer has been proved by [20] to coincide with $K^*_{G,\tau}(G)$. While we do not know if there is a reference of the WZW Verlinde algebra computation which conforms fully with mathematical standards of rigor, from a foundational point of view, the existence of a modular tensor category is the deeper question; the treatment of the fusion rules computation in the physics literature using singular vectors in the discrete series Verma modules over Kac-Moody algebra is, in our opinion, essentially correct.

By those computations, then, we *know* that the twisted K-theory realization of the projective modular functor associated with the chiral WZW models is the parametrized spectrum $(K_{G,\tau}(G))^G$ over $K(\mathbb{Z}, 3)$, and furthermore we know that the up to homotopy, the composition product and unit given by Theorem 2 coincides with the product constructed in [20].

Still, it would be nice to have an even more direct geometrical connection. For example, the homotopical interpretation of the Verlinde algebra product and unit [20] was also shown by the Kriz and Westerland [33] to relate to the product and unit of a twisted K-theory version of Chas-Sullivan's string topology [8]. Surprisingly, however, it was shown in [33] that the coproduct in twisted K-theory string topology does *not* give the right answer for a coproduct coming from a modular functor, and there is no augmentation in string topology at all. Therefore, perhaps the first operation in $(K_{G,\tau}(G))^G$ one should try to find a purely topological description of is the augmentation. As far as we know, no such description is known.

To conclude this section, we say a few words about the classification of the possible topological twistings (22). To talk about classification, we must introduce a notion of equivalence of topological twistings. This, however, is implicit in what we already said: A topological twisting (22) is a 2-weak multifunctor, which can be considered a weak multisorted algebra of operadic type where the variables are images of 2-morphisms of $\mathcal{A}^{top}_{spin}$. We therefore have a notion of a weak morphism of such structures (see 9.2). We call two topological twistings *equivalent* if a weak morphism exists between them. (Since the target is $\mathbb{C}^\times$, a weak morphism in the opposite direction automatically exists, too.) Note that the data specifying an equivalence of topological twistings will then be a map $h : \mathcal{A}_{spin}^{top} \to O^2\mathbb{C}^\times$.

Recall (a fact from [50] which is readily reproduced in our present formalism) that a modular functor determines a holomorphic $\mathbb{C}^\times$-central extension of the semigroup of annuli. Recall further from [50] that those $\mathbb{C}^\times$-central extensions are classified by a single complex number called the *central charge*. (This is the same number classifying the $\mathbb{C}$-central extensions of the Witt algebra of polynomial complex vector fields on $S^1$, although a formal passage between both contexts requires some technical care due to the fact that we are dealing with infinitely many dimensions.) In fact, one must prove that the central charge does not depend on label, but this can be done by taking an annulus with a given label, and cutting out a disk (which must have unit label). Comparing the variations of the different boundary components of the resulting pair of pants shows that any label has the same central charge as the unit label.

We then have the following

**Proposition 3.** *The topological twisting of a modular functor is, up to equivalence, completely determined by its central charge. The central charges of invertible modular functors are integral multiples of* $1/2$*. The central charges of modular functors which have trivial topological twisting (up to equivalence) are precisely the multiples of* $4$*.*

*Proof.* The first statement follows completely from the proof of Lemma 1, since the modular functor on the annuli we glue in is a line bundle $L$ determined, by definition, by the central charge. Over an object of $\mathcal{G}$, the bundle given by the modular functor is trivialized after tensoring with $L$. Therefore, modular functors with the same central charge produce equivalent data.

The second statement was proved in [32].

For the third statement, note from [32] that central charge 4 is realized by (the inverse of) the square of the Quillen determinant. This modular functor takes values only in (even) lines. Therefore, the topological twisting data are trivial, since the modular functor itself is a multifunctor $\mathcal{A}_{spin}^{top} \to O^2\mathbb{C}^\times$, whose topological twisting is by definition 0.

On the other hand, if the topological twisting of a modular functor is trivial up to equivalence, then, by definition, the modular functor becomes topological (i.e. acquires central charge 0) after being tensored with an invertible modular functor which takes values only in even lines. It follows from the classification in [32] that such invertible modular functors are isomorphic to even powers of the Quillen determinant.

□

**Comment:** Proposition 3 is interesting in part because we have produced a topological invariant which characterizes the central charge *modulo a certain number*. New speculations [4] about possible use of powers of the fermion conformal field theory for a geometric construction elliptic cohomology predict such a phenomenon (although we do not expect here to recover the exact periodicity of topological modular forms, in part due to the fact we omitted real structure and other refinements). It should be pointed out, however, that the formalism of modular tensor categories treated for example in [3] also gives an exponential of an imaginary multiple the central charge as a "topological invariant", namely the data contained in the modular tensor categories. We do not know whether modular tensor categories can be extended into a formalism which would fully determine the stack data of a holomorphic modular functor, and the central charge. This is why in the examples constructed in the next two sections, we will restrict attention to projective (Clifford) modular functors.

## 7. Clifford modular tensor categories and projective modular functors

In this Section, we shall discuss a method for constructing modular functors in the generality involving Clifford algebras. For simplicity, we will restrict attention to projective modular functors, which are sufficient for constructing the twisted K-theory realizations discussed in the last Section. We only know how to do this in a somewhat roundabout way. The fact is that currently, direct constructions of modular functors out of analytical data assigned to a Riemann surface, such as in the case of the chiral fermion [32], are generally unknown. The best known results on rigorous constructions of projective modular functors come from modular tensor categories, using the vertex operator algebra, and Huang's theorem [26].

In this section, we will discuss how to apply these methods in the Clifford case. Perhaps surprisingly, we will not define a concept of a "modular tensor category with spin". One reason is that to use such a notion, one would have to develop a separate discussion of Moore-Seiberg type constraints [3], Chapter 5.2 for Riemann surfaces with

Spin structure. Another reason is that in the Clifford modular case, in some cases, the s-matrix corresponding to an elliptic curve with Kervaire invariant 1 (i.e. on which every non-separating simple closed curve is periodic) is singular (in the case of the chiral fermion, this $1\times 1$ matrix is 0). We will explain what causes this "paradox", and how to get around it, a little later on.

It turns out that instead, projective Clifford modular functors come from ordinary modular tensor categories with certain extra structure. For the definition of a modular tensor category, we refer the reader to [3].

**Definition:** A *pre-Clifford modular tensor category* is a modular tensor category $C$ with product $\boxtimes$ and an object $V^-$ together with an isomorphism

$$\iota : V^- \boxtimes V^- \xrightarrow{\cong} 1 \tag{24}$$

which satisfies

$$\theta_{V^-} = -1. \tag{25}$$

(Note that (24) implies $(\theta_{V^-})^2 = 1$.)

In what follows, we will always assume that we are in a pre-Clifford modular tensor category as described in the definition.

**Lemma 4.** *We have* $(\sigma_{V^-V^-})^2 = 1$.

*Proof.* Compute:

$$\begin{aligned}&1 = \theta_{V^-\boxtimes V^-} = \sigma_{V^-V^-}\sigma_{V^-V^-}(\theta_{V^-}\otimes\theta_{V^-})\\ &= (\sigma_{V^-V^-})^2.\end{aligned}$$

□

**Lemma 5.** *Let* $M \in Obj(C)$. *Put*

$$\zeta_M = \sigma_{V^-M}\sigma_{MV^-}.$$

*Then* $(\zeta_M)^2 = 1$.

*Proof.* By the braiding relation, we have

$$\begin{aligned}&\sigma_{V^-(V^-\boxtimes M)}\sigma_{(V^-\boxtimes M)V^-}\\ &= (\sigma_{V^-V^-}\boxtimes Id_M)(Id_{V^-}\boxtimes\sigma_{V^-M}\sigma_{MV^-})(\sigma_{V^-V^-}\boxtimes Id)\\ &= Id_{V^-}\boxtimes\sigma_{V^-M}\sigma_{MV^-}.\end{aligned} \tag{26}$$

Now compute:

$$
\begin{aligned}
&1 \boxtimes 1 \boxtimes \theta_M = \theta_{V^- \boxtimes V^- \boxtimes M} \\
&= \sigma_{V^-(V^- \boxtimes M)} \sigma_{(V^- \boxtimes M)V^-} (\theta_{V^-} \boxtimes \theta_{V^- \boxtimes M}) \\
&= \sigma_{V^-(V^- \boxtimes M)} \sigma_{(V^- \boxtimes M)V^-} (\theta_{V^-} \boxtimes (\sigma_{V^- M} \sigma_{M V^-} (\theta_{V^-} \boxtimes \theta_M))) \\
&= (1 \boxtimes \zeta_M)(1 \boxtimes \zeta_M \theta_M) \\
&= \theta^2_{V^-} \boxtimes \zeta^2_M (1 \boxtimes \theta_M) = 1 \boxtimes \zeta^2_M (1 \boxtimes \theta_M).
\end{aligned}
$$

Thus, $\zeta^2_M = Id_{V^- \boxtimes M}$, as claimed. □

If $M$ is irreducible, then $\zeta_M \in \mathbb{C}^\times$, so $\zeta_M \in \{\pm 1\}$.

**Lemma 6.** *Let $M, N \in Obj(C)$ be irreducible. Then*

$$\zeta_{M \boxtimes N} = \zeta_M \cdot \zeta_N.$$

*Proof.* Using the braiding relation,

$$
\begin{aligned}
&\sigma_{V^-(M \boxtimes N)} \sigma_{(M \boxtimes N)V^-} \\
&= (\sigma_{V^- N} \boxtimes Id_N)(Id_M \boxtimes \sigma_{V^- N} \sigma_{N V^-})(\sigma_{M V^-} \boxtimes Id_N) \\
&= \zeta_N (\sigma_{V^- M} \boxtimes Id_N)(\sigma_{M V^-} \boxtimes Id_N) = \zeta_N \zeta_M.
\end{aligned}
$$

□

We call an irreducible object $M$ *Neveu-Schwarz (or NS)* (resp. *Ramond (or R)*) if $\zeta_M = 1$ (resp. $\zeta_M = -1$).

**Lemma 7.** *An irreducible object is NS (resp. R) if and only if*

$$\theta_{V^- \boxtimes M} = -Id_{V^-} \boxtimes \theta_M$$

*resp.*

$$\theta_{V^- \boxtimes M} = Id_{V^-} \boxtimes \theta_M.$$

*Proof.* We have

$$\theta_{V^- \boxtimes M} = \zeta_M (\theta_{V^-} \boxtimes \theta_M) = -\zeta_M (Id_{V^-} \boxtimes \theta_M).$$

□

Next, we will make some observations on counting isomorphism classes of irreducible R and NS objects (also called *labels*), and the $s$-matrix. To this end, we need some additional notation. Note that $V^-\boxtimes ?$ defines an involution $\overline{?}$ of isomorphism classes of irreducible objects of $C$. The fixed points of the involution are all R. We denote the set of fixed points by $R^0$, and call them *non-split R labels.* The regular orbits can consist of NS or R labels. Choose a set $NS^+$ of representatives of regular NS orbits, and a set of representatives $R^+$ of R orbits. Let also $NS^- = \overline{NS^+}$, $R^- = \overline{R^+}$. The elements of $NS = NS^+ \cup NS^-$ will be called *NS labels*, the elements of $R^\pm = R^+ \cup R^-$ *split R labels.*

**Lemma 8.** *(1) Let $i$ be a label. Then*

$$s_{\bar{i}j} = \begin{array}{rl} s_{ij} & \textit{if } j \in NS \\ -s_{ij} & \textit{if } j \in R. \end{array} \tag{27}$$

*(2) If $i \in R_0, j \in R$, then*

$$s_{ij} = 0. \tag{28}$$

*Proof.* We have

$$s_{ij} = \theta_i^{-1}\theta_j^{-1}\sum_k N_{ij}^k \theta_k d_k \tag{29}$$

where $d_k$ is the quantum dimension (see [3]). We have

$$N_{\bar{i}j}^{\bar{k}} = N_{ij}^k,$$

and also

$$d_{\overline{k}} = d_k \cdot d_{V^-} = d_k,$$

since $V^-$ is invertible and hence its quantum dimension is 1 ([13]). Also, if $j \in NS$, then $i, k$ are both NS or both R by Lemma 6. Thus,

$$\theta_i\theta_k = \theta_{\bar{i}}\theta_{\overline{k}}.$$

Similary, if $j \in R$, then by Lemma 6, one of the labels $i, k$ is NS and the other is R. Thus,

$$\theta_i\theta_k = -\theta_{\bar{i}}\theta_{\overline{k}}.$$

Consequently, (27) follows from (29). To prove (28), just note that for $i \in R_0$, $j \in R$, by (27), we have $\overline{i} = i$, so

$$s_{ij} = s_{\bar{i}j} = -s_{ij}.$$

$\square$

By Lemma 8, classified by the type of labels, the s-matrix has the following form:

(30)

| | $NS^+$ | $NS^-$ | $R^+$ | $R^-$ | $R^0$ |
|---|---|---|---|---|---|
| $NS^+$ | $A$ | $A$ | $B$ | $B$ | $D$ |
| $NS^-$ | $A$ | $A$ | $-B$ | $-B$ | $-D$ |
| $R^+$ | $B^T$ | $-B^T$ | $C$ | $-C$ | $0$ |
| $R^-$ | $B^T$ | $-B^T$ | $-C$ | $C$ | $0$ |
| $R^0$ | $D^T$ | $-D^T$ | $0$ | $0$ | $0$ |

**Proposition 9.** *The matrices $A$ and $C$ of Table 30 are symmetric and non-singular. The matrix $\begin{pmatrix} B & D \end{pmatrix}$ is non-singular, and we have*

$$B^T D = 0. \tag{31}$$

*We also have*

$$|R^+| + |R^0| = |NS^+|. \tag{32}$$

*Proof.* The s-matrix is symmetrical, hence so are the matrices $A$, $C$. By performing row and column operations on the matrix (30), we may obtain the matrix

$$\begin{pmatrix} 0 & 0 & 0 & B & D \\ 0 & 4A & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 \\ B^T & 0 & 0 & 0 & 0 \\ D^T & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, the statements about non-singularity of matrices follow, since the s-matrix is non-singular. Therefore, (32) follows. To prove (31), recall that the square of the s-matrix is a scalar multiple of the charge conjugation matrix, and note that obviously, $R^0$ and $R^\pm$ are invariant under the operation of taking contragredient labels. $\square$

**Lemma 10.** *The following diagram commutes:*

$$\begin{tikzcd} V^- \boxtimes V^- \boxtimes V^- \arrow[r, "\iota\boxtimes 1", "\cong"'] \arrow[d, "1\boxtimes\iota"', "\cong"] & 1 \boxtimes V^- \arrow[d, "\cong"', "\eta"] \\ V^- \boxtimes 1 \arrow[r, "\cong", "\eta"'] & V^- \end{tikzcd}$$

*where $\eta$ is the unit coherence isomorphism.*

*Proof.* Recall that by Lemma 4 and irreducibility, $\sigma_{V^-V^-} = \pm 1$. Thus, by the braiding relation,

$$\eta(\iota \boxtimes 1) = \eta(1 \boxtimes \eta)(\sigma_{V^-V^-} \boxtimes 1)(1 \boxtimes \sigma_{V^-V^-}) = \eta(1 \boxtimes \iota).$$

□

**Lemma 11.** *Let $X \in R^0$. Then there exists an isomorphism*

$$\theta : V^- \boxtimes X \to X$$

*so that the following diagram commutes:*

$$\begin{tikzcd} V^- \boxtimes V^- \boxtimes X \arrow[r, "1\boxtimes\theta"] \arrow[d, "\iota"'] & V^- \boxtimes X \arrow[d, "-\theta"] \\ 1 \boxtimes X \arrow[r, "\eta"'] & X. \end{tikzcd}$$

*Proof.* By irreducibility, the diagram commutes up to multiplication by a non-zero complex number $\lambda$. Hence, it suffices to replace $\theta$ by $\theta/\sqrt{\lambda}$. □

From now on, we will assume that a choice of $\theta$ has been made as in Lemma 11.

**Definition:** A pre-Clifford modular tensor category is called a *Clifford tensor category* if

$$\sigma_{V^-V^-} = -1. \tag{33}$$

**Remarks:** 1. We do not know whether there exists pre-Clifford modular tensor categories in which both $R^0 \neq \emptyset$ and $R^\pm \neq \emptyset$.

2. In the remainder of this Section, we will produce a Clifford projective modular functor from a Clifford tensor category. The assumption (33) is essential to our arguments. The alternative to (33) is, by Lemma 4,

$$\sigma_{V^-V^-} = 1. \tag{34}$$

In the paper [32], there naturally appeared, as an alternative to Spin structure on Riemann surfaces something called the "Sqrt structure", which is, vaguely speaking, an "untwisted analogue" of Spin structure. We believe that replacing (33) with (34) could be used to produce a notion of a modular functor on Riemann surfaces with Sqrt structure, but do not follow this direction in detail, since it appears to be currently of lesser importance from the point of view of examples.

The remainder of this section from this point on is dedicated to constructing a projective Clifford modular functor on the weak $\star$-category of Riemann surfaces with Spin structure from a Clifford modular tensor category $C$. Of course $C$, being a modular tensor category, by the construction of [3], Chapter 5, in particular defines an ordinary projective modular functor $M$ on the weak $\star$-category of Riemann surfaces (without Spin structure). This phenomenon is well known in mathematical physics. For example, in [21], the modular functor $M$ is referred to as the *spin model*. Later, it became more widely known as the *GSO projection*.

We denote by $S(X_1, \dots, X_n)$ a standard sphere ([3], Section 5.2) with $n$ punctures $\{1, \dots, n\}$ oriented outbound, labelled by $n$ irreducible objects $X_1, \dots, X_n \in Obj(C)$. Denote, for an irreducible object $X \in Obj(C)$, $\epsilon \in \mathbb{Z}/2$,

$$X(\epsilon) = \begin{array}{rl} X & \text{if } \epsilon = 0 \\ V^- \boxtimes X & \text{if } \epsilon = 1. \end{array}$$

Put

$$\begin{aligned}
&\widetilde{M}(S(X_1, \dots, X_n)) \\
&= \bigoplus_{\substack{\epsilon_i \in \mathbb{Z}/2 \\ i = 1, \dots, n}} M(X_1(\epsilon_1), \dots, X_n(\epsilon_n)) \\
&= \bigoplus_{\substack{\epsilon_i \in \mathbb{Z}/2 \\ i = 1, \dots, n}} Hom_C(1, X_1(\epsilon_1) \boxtimes \dots \boxtimes X_n(\epsilon_n))).
\end{aligned}$$

We may consider $\widetilde{M}(X_1, \dots, X_n)$ as a $(\mathbb{Z}/2)^n$-graded vector space by $(\epsilon_1, \dots, \epsilon_n)$. We may also consider a "total" $\mathbb{Z}/2$-grading by $\epsilon_1 + \dots + \epsilon_n$. Denote by $1 \leq i_1 < \dots < i_k \leq n$ all those indices $i$ such that $X_i$ We will construct commuting involutions $\alpha_i$, $i = 1, \dots, n-1$ of $(\mathbb{Z}/2)^n$-degree

$$(\underbrace{0, 0, \dots, 0}_{i-1}, 1, 1, 0, \dots, 0),$$

and *anticommuting* involutions $\lambda_j$ of $(\mathbb{Z}/2)^n$-degree

$$(\underbrace{0,0,\dots,0}_{i_j-1},1,0,\dots,0),$$

where $\alpha_i$, $\beta_j$ commute for any $i=1,\dots,n$, $j=1,\dots,k$.

The operator $\alpha_i$ is of the form

$$\bigoplus Hom_C(1,\\ Id_{X_1(\epsilon_1)}\boxtimes\dots\boxtimes Id_{X_{i-1}(\epsilon_{i-1})}\boxtimes q\boxtimes Id_{X_{i+2}(\epsilon_{i+2})}\boxtimes\dots\boxtimes Id_{X_n(\epsilon_n)})$$

where

$$q: X(\epsilon_1)\boxtimes Y(\epsilon_2)\to X(\epsilon_1+1)\boxtimes Y(\epsilon_2+1)$$

is given, according to the different values of $\epsilon_1,\epsilon_2\in\mathbb{Z}/2$, as follows:

$$\begin{tikzcd}
X\boxtimes Y \arrow[rr, "q"] \arrow[dr, "\iota^{-1}\boxtimes 1"'] & & V^-\boxtimes X\boxtimes V^-\boxtimes Y \\
 & V^-\boxtimes V^-\boxtimes X\boxtimes Y \arrow[ur, "1\boxtimes\sigma_{V^-X}\boxtimes 1"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
X\boxtimes V^-\boxtimes Y \arrow[r, "q"] \arrow[d, "\iota^{-1}\boxtimes 1\boxtimes 1"'] & V^-\boxtimes X\boxtimes Y \\
V^-\boxtimes V^-\boxtimes X\boxtimes V^-\boxtimes Y \arrow[r, "1\boxtimes\sigma_{V^-X}\boxtimes 1\boxtimes 1"'] & V^-\boxtimes X\boxtimes V^-\boxtimes V^-\boxtimes Y \arrow[u, "1\boxtimes 1\boxtimes\iota\boxtimes 1"']
\end{tikzcd}$$

$$V^-\boxtimes X\boxtimes Y \xrightarrow{q=\sigma_{V^-X}\boxtimes 1} V\boxtimes V^-\boxtimes Y$$

$$\begin{tikzcd}
V^-\boxtimes X\boxtimes V^-\boxtimes Y \arrow[rr, "q"] \arrow[dr, "\sigma_{V^-X}\boxtimes 1\boxtimes 1"'] & & X\boxtimes Y \\
 & X\boxtimes V^-\boxtimes V^-\boxtimes Y. \arrow[ur, "1\boxtimes\iota\boxtimes 1"'] &
\end{tikzcd}$$

The operator $\lambda_j$ is

$$\bigoplus(-1)^{(\sum_{s=1}^{i_j-1}\epsilon_s)}Hom_C(1,\underbrace{1\boxtimes\dots\boxtimes 1}_{i_j-1\text{ times}}\boxtimes\lambda\boxtimes 1\boxtimes\dots\boxtimes 1)$$

where for $X\in R^0$, $\epsilon\in\mathbb{Z}/2$,

$$\lambda: X(\epsilon)\to X(\epsilon+1)$$

is given as follows:

$$V^-\boxtimes X\xrightarrow{\lambda=\theta} X$$

$$\begin{tikzcd} X \arrow[r, "\lambda=\theta^{-1}"] \arrow[d, "\alpha\boxtimes 1"'] & V^- \boxtimes X \\ V^- \boxtimes V^- \boxtimes X \arrow[r, "\sigma_{V^-V^-\boxtimes X}"'] & V^- \boxtimes V^- \boxtimes X \arrow[u, "1\boxtimes\theta"'] \end{tikzcd}$$

(this diagram commutes by Lemma 11.

**Lemma 12.** *The operators $\iota_i$, $i = 1 \dots n$ commute and satisfy $(\iota_i)^2 = 1$. The operators $\lambda_j$, $j = 1, \dots, k$ anticommute and satisfy $(\lambda_j)^2 = 1$. Moreover, every operator $\alpha_i$ commutes with every operator $\lambda_j$.*

*Proof.* A straightforward computation using Lemmas 10, 11 and the braiding relation. □

Now note that each of the involutions $\alpha_i$, since it is $\mathbb{Z}/2$-graded of odd degree with respect to grading by the $i$'th copy of $\mathbb{Z}/2$, is diagonalizable, and half of its eigenvalues are $+1$, half are $-1$. Moreover, since $\alpha_i$ commute, they are simultaneously diagonalizable.

Moreover, each $\alpha_1 \times \cdots \times \alpha_{n-1}$-weight $(w_1, \dots, w_{n-1}) \in (\mathbb{Z}/2)^{n-1}$ corresponds to a spin structure on $S(X_1, \dots, X_n)$ as follows: On the boundary circle of the $i$'th puncture, put an antiperiodic (resp. periodic) Spin structure depending on whether $X_i$ is NS or R. Furthermore, identify the spinors on all of the points $P_1, \dots, P_n$ of each of the circle of lowest imaginary part with $\mathbb{R}$. On the path from $P_i$ to $P_{i+1}$ along the marking graph of $S(X_1, \dots, X_n)$ (see [3], Section 5.2), put the antiperiodic resp. periodic Spin structure depending on whether $w_i = -1$ or $w_i = 1$. Denote the resulting spin structure on $S(X_1, \dots, X_n)$ by $\sigma(w_1, \dots, w_n)$.

Let

$$M(S(X_1, \dots, X_n), \sigma(w_1, \dots, w_{n-1}))$$

be the $(w_1, \dots, w_{n-1})$ weight space of

$$\widetilde{M}(S(X_1, \dots, X_n), \sigma(w_1, \dots, w_{n-1}))$$

with respect to $(\alpha_1, \dots, \alpha_{n-1})$, considered as a left module over

$$\Lambda = T_{\mathbb{C}}(\lambda_1, \dots, \lambda_j)/(\lambda_i\lambda_k = -\lambda_k\lambda_i, \lambda_i^2 = 1) \tag{35}$$

(where $T_{\mathbb{C}}$ denotes the $\mathbb{C}$-tensor algebra on the given generators).

In discussing the passage from a modular tensor category to a modular functor, [3] do not discuss orientation of boundary components in detail. This is because reversal of orientation of a boundary component can be always accomplished by changing a label to its contragredient label.

In the Spin case, however, we need to be more careful because reversal of orientation of a periodic boundary component does not carry a canonical Spin structure: A cylinder with two inbound (or two outbound) boundary components has two different possible Spin structures which are interchanged by a diffeomorphism interchanging te boundary components.

To discuss reversal of orientation, recall that in the standard sphere $S(X_1, \dots, X_n)$ with a spin structure $\sigma = \sigma(w_1, \dots, w_{n-1})$, the boundary components decorated by the labels $X_1, \dots, X_{n-1}$ were oriented outbound. We may create a *mirror* sphere $\overline{S}(X_1^*, \dots, X_n^*)$ (together with a canonical Spin structure $\overline{\sigma}$) by reflecting by the imaginary axis, and replacing labels by contragredient ones. (Note: To facilitate gluing, we only need to consider $n = 2$.)

Let

$$M(\overline{S}(X_1^*, \dots, X_n^*), \overline{\sigma}) = Hom_\Lambda(M(S(X_1, \dots, X_n), \sigma), \Lambda)$$

where $\Lambda$ is the Clifford algebra (35). Thus, $M(\overline{S}(X_1^*, \dots, X_n^*), \overline{\sigma})$ is naturally a *right* $\Lambda$-module, hence a left $\Lambda^{Op}$-module). Note that $\Lambda^{Op}$ is isomorphic to $\Lambda$, but not canonically. In fact, using the Koszul signs, we have, canonically,

$$(T_{\mathbb{C}}(\alpha)/\alpha^2 = 1)^{Op} = T_{\mathbb{C}}(\alpha^*)/((\alpha^*)^2 = -1).$$

However, for every $\mathbb{C}$-algebra $A$, $A \otimes A^{Op}$ has a canonical bimodule (naturally identified with $A$) from either side, and applying this to $\Lambda$ facilitates gluing of an inbound and outbound boundary component with Spin structure.

Now the effect of gluing and moves on $M(\Sigma, \sigma)$ for labelled surfaces $\Sigma$ with spin structure $\sigma$ (up to scalar multiple) follows from the corresponding statement on the GSO projection, taking into account the change of Spin structure caused by the move. (This is why we don't need a separate "lego game" for surfaces with Spin structure.) All the statements are straightforward consequences of the definition, and we omit the details.

One case, however, warrants special discussion, namely the S-move. We have proved above in Proposition 9 that the s-matrices corresponding to elliptic curves of Kervaire invariant 0 (NS-NS and NS-R) are non-singular. In the case of the elliptic curve of Kervaire invariant 1 (R-R), we only know that the $R^+$-$R^+$ s-matrix is non-singular, while the rest of the s-matrix is zero!

To explain this effect, note that when gluing

$$S(X_1, X_2), \sigma$$

to

$$\overline{S}(X_1^*, X_2^*), \overline{\sigma}$$

where $X_1, X_2$ are non-split Ramond (note: we necessarily have $X_1 = X_2^*$), the curve spanned by the two marking graphs of $S(X_1, X_2), \sigma$ and $\overline{S}(X_1^*, X_2^*), \overline{\sigma}$ is antiperiodic, since we are gluing the boundary at the angle $\pi$ and not 0. The trace, in this case ,then, is a copy of $\mathbb{C}$ for each $R_0$ label: this is the R-NS elliptic curve).

To obtain the R-R curve, we replace one of the Spin structures, say, $\overline{\sigma}$, with the other possible Spin structure on $\overline{S}(X_1^*, X_2^*)$. This results in the *reversal of signs* of the action of one of the generators $\Lambda_1$ or $\Lambda_2$ (depending on how exactly we identify the spinors at the points $P_1, P_2$, which is also non-canonical).

In any case, one readily verifies that if we take the trace after this modification of Spin structure, we get 0. Thus, the vector space assigned by our construction to the Kervaire invariant 1 elliptic curve is, in fact, the free $\mathbb{C}$-moduls on the set of $R^+$-labels!

**Remark:** There are examples of Clifford modular tensor categories with split ($R^{\pm}$) labels (for example an even power of the chiral fermion), and examples of Clifford modular tensor categories with non-split ($R^0$) labels (for example an odd power of the chiral fermion). We do not know, however, an example of a Clifford modular tensor category which would have both split and non-split R labels.

## 8. Super vertex algebras, $N = 1$ supersymmetric minimal models.

In this section, we will describe how Clifford modular tensor categories (and hence projective Clifford modular functors) may be obtained from super vertex algebras, and we will specifically discuss the example of $N = 1$ supersymmetric minimal models. For a definition of a super vertex algebra, and basic facts about this concept, we refer the reader to Kac [31]. In this paper, we will only consider (super) vertex algebras with a conformal element $L$ (also denoted by $\omega$, cf. [31]). For a super vertex algebra $V$, we denote by $V^+$ (resp. $V^-$) the submodule of elements of weights in $\mathbb{Z}$ (resp. $(1/2) + \mathbb{Z}$).

**Theorem 13.** *Let $V$ be a super-vertex algebra with a conformal element $L$ which satisfies Huang's conditions* [26]*:*

*(1) $V_{<0} = 0$, $V_0 = \mathbb{C}$ and the contragredient module to $V$ is $V$.*

*(2) Every $V$-module is completely reducible.*

*(3) $V$ satisfies the $C_2$-condition. (Explicitly, the quotient of $V$ by the sum of the images of $aV$ where $a$ is a coefficient of $z^{\geq 1}$ of the vertex operator $Y(u,z)$, $u \in V$, is finite-dimensional.)*

*Then the category of finitely generated $V^+$-modules is a Clifford modular tensor category (and consequently, by the construction of the last Section produces an example of a projective Clifford modular functor.)*

*Proof.* Clearly, condition (1) passes on to $V^+$. Conditions (2) and (3) pass on to $V^+$ by the results of Miyamoto [41, 42, 43, 44], applying them to the case of the $\mathbb{Z}/2$ acting on $V$ by 1 on $V^+$ and $-1$ on $V^-$. (While Miyamoto does not discus super vertex algebras explicitly, his arguments are unaffected by the generalization.)

Thus, it remains to prove that $V^- \boxtimes V^- \cong V^+$ in the category of $V^+$-modules. The super vertex algebra structure gives a canonical map

$$V^- \boxtimes V^- \xrightarrow{\mu} V^+. \tag{36}$$

The map must be onto since $V^+$ is an irreducible $V^+$-modules, and if the image of (36) were 0, $V^-$ would be an ideal in $V$.

Suppose $\mu$ is not injective. Let $M = Ker(\mu)$. Note that any $V^+$-module $X$ with a map $V^- \boxtimes X \to X$ which satisfies associativity with the $V^+$-module structure and the map and the map (36) is a weak $V$-module. (In this proof, $\boxtimes$ means the fusion tensor product in the category of $V^+$-modules.) Thus, in particular,

$$V \boxtimes V^-$$

is a weak $V$-module, and the map

$$\phi : V \boxtimes V^- \to V$$

given by right multiplication by the $V^+$-module $V^-$ is a map of weak $V^+$-modules. Consequently, $M = Ker(\phi)$ is a weak $V^+$-module and hence, by dimensional considerations, a $V^+$-module. Also for dimensional reasons, $V^-$ annihilates the $V$-module $M$. Hence, the $V$-annihilator of $M$ is a non-trivial ideal in $V$, which is a contradiction.

Thus, $M = 0$ and $\mu$ is injective. □

**Example:** The $N = 1$ supersymmetric minimal model is a super vertex algebra obtained as a quotient $L_{p,q}$ of the Verma module $V(c_{p,q}, 0)$ of the $N = 1$ Neveu Schwarz algebra $\mathcal{A}$ (for a definition, see e.g. [23])

by the maximal ideal, where

$$c_{p,q} = \frac{3}{2}\left(1 - \frac{2(p-q)^2}{pq}\right), \tag{37}$$

$p, q \in \mathbb{Z}_{\geq 2}$, $p \equiv q \mod 2$ and $gcd(p,(p-q)/2) = 1$. Non-isomorphic irreducible NS (resp. R) modules are given by the $N = 1$ minimal models $L_{p,q,r,s}$ with central charge $c$, i.e. quotients of the Verma module $V(c_{p,q}, h_{r,s})$ over the NS (resp. R) algebra (for a definition of the R-algebra, see e.g. [28]) where $1 \leq r \leq p-1$, $1 \leq r \leq q-1$, $r, s \in \mathbb{Z}$ and $r \equiv s \mod 2$ (resp. $r+1 \equiv s \mod 2$) and

$$h_{r,s} = \frac{(pr-qs)^2 - (p-q)^2}{8pq} + \frac{\epsilon}{16}$$

where $\epsilon = 0$ (resp. $\epsilon = 1$). Additionally, we have

$$L_{p,q,r,s} \cong L_{p,q,p-r,q-s}$$

(which, for dimensional reasons, are the only possible isomoprhisms between these irreducible modules). Thus, when both $p, q$ are odd, there are

$$\frac{(p-1)(q-1)}{4} \text{ NS (resp. R) irreducible modules,}$$

and when $p, q$ are both even (in which case $p - q \equiv 2 \mod 4$), there are

$$\frac{(p-1)(q-1)+1}{4} \text{ NS (resp. R) irreducible modules.}$$

By a result of Zhu [53], a vertex algebra $V$ cannot have more irreducible modules than the dimension of $V/C_2V$. Furthermore, when equality arises, the Zhu algebra is a product of copies of $\mathbb{C}$, and hence is semisimple. This is the case of $V = L_{p,q}$ by Theorem 16. Hence, we also know that the above list of irreducible modules $L_{p,q,r,s}$ is complete.

To prove the condition of complete reducibility (condition (2) of Theorem 13), it then suffices to show that

$$Ext^1(M, N) = 0$$

for any two irreducible modules. This is proved in [23] for the case of NS modules. Since there can only be non-trivial extensions if $M, N$ are both NS or both R, assume that $M, N$ are both R (the argument we are about to give works in both cases). Let $\mathcal{A}^R_-$ be the subalgebra of the Ramond algebra $\mathcal{A}^R$ spanned by $G_{\geq 0}$, $L_{\geq 0}$. Then we have a BGG resolution of $M$ by Verma modules

$$V_h = \mathcal{A}^R \otimes_{\mathcal{A}^R_-} V_h^0$$

where on $V_h^0$, $\mathcal{A}_-^R$ acts through its 0 degree, and $V_h^0$ is 2-dimensional, with $L_0$ acting by $h$, and $G_0$ acting by $\pm\sqrt{h}$ on the two basis elements.

Then the BGG resolution of $M$ has the form

$$\ldots \to \bigoplus_k V_{h_{i_2},k} \to \bigoplus_k V_{h_{i_1},k} \to V_h,$$
$$h_{i_j,k} > h,\ h_{i_j,k} \in h + \mathbb{Z}.$$

This leads to a spectral sequence

$$E_1^{p,q} = \prod_k Ext^q_{\mathcal{A}_-^R}(V_{h_{i_p},k}, N) \Rightarrow Ext^{p+q}_{\mathcal{A}^R}(M, N). \tag{38}$$

For dimensional (integrality) reasons, (38) can only be non-zero for $M = N$ and the only terms we need to worry about are $p = 0, q = 1$ and $p = 1, q = 0$. The former is excluded by the fact that $h$ is the lowest weight of $M = N$, the latter by the fact that $N$ has no singular vectors.

Thus, the asumptions are verified, and the $N = 1$ supersymmetric minimal models give examples to which Theorem 13 applies. Note that by Lemma 15, (the parity of the number of $G$'s), it follows that all the R labels are split when $(p-1)(q-1)$ is even, and non-split when $(p-1)(q-1)$ is odd.

## 9. Appendix: Some technical results

We shall describe here some constructions needed to make rigorous the results of the previous sections. We will start with May-Thomason rectification.

9.1. **Rectification of weak multicategories.** The idea is to approach the problem much more generally. A *universal algebra of operadic type* $\mathcal{T}$ is allowed to have any set $I$ (possibly infinite) of $n_i$-ary operations $\circ_i$ ($n_i \geq 0$ finite), $i \in I$ and any set $J$ (possibly infinite) of *relations* of the form

$$w_j(x_1, \ldots, x_{m_j}) = w'_j(x_1, \ldots, x_{m_j}),\ j \in J \tag{39}$$

where $w_j$, $w'_j$ are finite "words" one can write using the different variables $x_1, \ldots, x_{m_j}$ and the operations $\circ_i$, such that on both sides of (39), every variable $x_1, \ldots, x_{m_j}$ occurs precisely once (some operations on the other hand may be repeated, or may not occur at all).

A $\mathcal{T}$-algebra then is a model of this universal algebra structure, i.e. a set with actually $n_i$-ary operations $\circ_i$, which satisfy the relations (39), when we plug in concrete (possibly repeating) elements for $x_1, \ldots, x_{m_j}$.

For a universal algebra of operadic type $\mathcal{T}$, there exists a canonical operad $\mathcal{C}_{\mathcal{T}}$ such that the category of $\mathcal{T}$-algebras is canonically equivalent to the category of $\mathcal{C}_{\mathcal{T}}$-algebras. In fact, if we introduce the smallest equivalence relation on words which is stable with respect to substitutions, and such that the left and right hand sides of (39) are equivalent words, then we have

$$\mathcal{C}_{\mathcal{T}}(n) = \{\text{equivalence classes of words in } x_1, \dots, x_n\} \tag{40}$$

We shall call the type $\mathcal{T}$ *free* if the operad $\Sigma_n$-action on $\mathcal{C}_{\mathcal{T}}$ is free.

If $\mathcal{T}$ is a universal algebra of operadic type, then a *weak $\mathcal{T}$-algebra* is a groupoid with functorial operations $\circ_i$, $i \in I$ where each equality (39) is replaced by a natural isomorphism (called *coherence isomorphism*). These coherence isomorphisms are required to form *coherence diagrams* which are described as follows: Suppose we have a sequence $w_0, \dots, w_m$, $w_m = w_0$ of words in non-repeating variables $x_1, \dots, x_n$, each of which is used exactly once, such that

$$\begin{array}{l} w_k(x_1, \dots, x_n) = w(\dots, w_{j_k}(\dots)\dots) \\ w_k(x_1, \dots, x_n) = w(\dots, w'_{j_k}(\dots)\dots), \end{array} \tag{41}$$

$k = 0, \dots, m-1$ (i.e. at each step, we make a change along (39), combined with substitutions (which means we can substitute into the variables inside the word, or the whole word may be used as a variable for further operations, as long as, again, every variable $x_1, \dots, x_n$ ends up used exactly once in the whole word). Then there is an obvious coherence diagram modelled on the sequence $w_1, \dots, w_m$.

If $\mathcal{T}$ is a free operadic type of universal algebras, then a weak $\mathcal{T}$-algebra can be rectified into a $\mathcal{T}$-algebra by the following construction due to May and Thomason [40]: First, consider the (strict) operad enriched in groupoids $\mathcal{C}'$ which is the free operad $\mathcal{O}_{\mathcal{C}}$ on the sequence of sets $(\mathcal{C}(n))_{n \geq 0}$, where we put precisely one isomorphism

$$x \cong x'$$

for $x, x' \in \mathcal{O}_{\mathcal{C}}(n)$ such that

$$\epsilon(x) = \epsilon(x') \in \mathcal{C}(n)$$

where $\epsilon : \mathcal{O}_{\mathcal{C}} \to \mathcal{C}$ is the counit of the adjunction between the forgetful functor from operads to sequences and the free operad functor.

Now denoting by $|\mathcal{C}'|(n)$ the nerve (bar construction) on $\mathcal{C}'(n)$, $\epsilon$ induces a map of operads

$$\iota : |\mathcal{C}'| \to \mathcal{C} \tag{42}$$

which is an equivalence on each $n$-level. If $\mathcal{T}$ is of free type, (42) induces an equivalence of monads

$$C' \to C$$

where $C$ is the monad associated with $\mathcal{C}$, and $C'$ is the monad associated with $|\mathcal{C}'|$; recall that for an operad $\mathcal{C}$, the associated monad is

$$C(X) = \coprod_{n\geq 0} \mathcal{C}(n) \times_{\Sigma_n} X^n. \tag{43}$$

Thus, the rectification of the nerve $|X|$ of a weak $\mathcal{C}$-algebra $X$ from a $|\mathcal{C}'|$-algebra to a $\mathcal{C}$-algebra is

$$B(C, C', X) \xleftarrow{\sim} \mathbb{B}(C', C', X) \xrightarrow{\sim} X.$$

Now the exact same discussion applies to *multisorted algebras of operadic type*. This means that there is an additional set $K$ of objects and each $i \in I$ comes with an $n_i$-tuple $(k_1, \dots, k_{n_i}) \in K^{n_i}$ of input objects, and an output object $\ell_i \in K$. An algebra of the type then consists of sets $X_k$, $k \in K$ and operations using elements of the prescribed input sets, and producing an element of the prescribed output set. Again, the operations must satisfy the prescribed relations, which are of type (39) (the only difference being that every $x_i$ is decorated with an object and one must keep track of objects when applying the operations).

The associated operad (40) then becomes a multisorted operad with objects $K$ (which is actually the same thing as a multicategory with objects $K$). The associated monad to a multisorted operad is in the category of $K$-tuples of sets, which can also be considered as sets fibered over $K$ (i.e. sets together with a map into $K$). Formula (43) is then correct with $\times$ replaced by $\times_K$, the fibered product over $K$.

The definition of free operadic type remains the same, with $\Sigma_n$ replaced by the isotropy group of a particular fibration $\{1, \dots, n\} \to K$.

Now for us, the main point is that multicategories with object set $B$ (over $Id : B \to B$) are a multisorted algebra of operadic type, where the set of objects is

$$\coprod_{n\geq 0} B^{n+1}.$$

Furthermore, this type is free (as is readily verified by inspection of the axioms). Hence, weak multicategories can be rectified into strict multicategories using the May-Thomason rectification.

9.2. **Rectification of weak multimorphisms.** A *weak morphism of weak $\mathcal{T}$-algebras* has the same data as a morphism (i.e. a map of sets preserving the operations), but instead for each operation, we have a coherence isomorphism. The coherence diagrams in this case are easier: there is one coherence diagram for each relation (39).

The construction described in the previous subsection is functorial, so it automatically rectifies a weak morphism to a morphism. (In the multi-sorted case, it even handles a function on objects.) In particular, a weak multifunctor between weak multicategories is rectified into a strict one.

Suppose now

$$F : \mathcal{A} \to \mathcal{C}$$

is a weak multifunctor where $\mathcal{A}$ is a weak multicategory and $\mathcal{C}$ is a strict multicategory enriched in groupoids. Then we should be entitled to more information (i.e. we should not be required to rectify the already strict multicategory $\mathcal{C}$). In effect, this works out. Using $C$ and $C'$ in the same meaning as above, we get a morphism of $C'$-algebras

$$B_2F : B_2\mathcal{A} \to B_2\mathcal{C}.$$

Hence, we obtain morphisms of $C$-algebras (i.e. multifunctors)

$$\begin{tikzcd}
B(C, C', B_2\mathcal{A}) \arrow[r, "B(C,C',B_2F)"] & B(C, C', B_2\mathcal{C}) \arrow[d, "B(C,\epsilon,B_2\mathcal{C})"] \\
& B(C, C, B_2\mathcal{C}) \arrow[d] \\
& B_2\mathcal{C}
\end{tikzcd}$$

where the last map is the usual simplicial contraction.

9.3. **The topological Joyal-Street construction.** In this paper we have to deal with categories where both the sets of objects and morphisms are topological spaces. The appropriate setting then is the notion of a *$T$-category*, by which we mean a category $C$ where both the sets of objects and morphisms are (compactly generated) topological spaces, $S, T : Mor(C) \to Ob(C)$ are fibrations and the unit and composition are continuous. In fact, for simplicity, let us assume that $C$ is a groupoid, by which we mean that there is an inverse operation $Mor(C) \to Mor(C)$ which is continuous. Then we shall speak of a $T$-groupoid.

Next, a *symmetric monoidal T-category* is a T-category $C$ with a continuous functor $\oplus : C \times C \to C$ (continuous on both objects and morphisms) and a unit $0 \in Obj(C)$ satisfying the usual axioms of a symmetric monoidal category, with the coherence natural transformations continuous (as maps $Obj(C) \to Mor(C)$). A *permutative T-category* is a symmetric monoidal T-category where $\oplus$ is strictly associative unital.

**Proposition 14.** *Let $C$ be a symmetric monoidal T-groupoid. Then there exists a permutative T-groupoid $C'$ and a continuous weakly symmetric monoidal functor (with continuous coherences)*

$$\Gamma : C \to C'$$

*which is a continuous equivalence of categories T-categories (i.e. has a continuous inverse where the compositions are continuously isomorphic to the identities).*

*Proof.* Just as in the classical case, the proof is exactly the same as when we replace "symmetric monoidal" by "monoidal" and "permutative" by "strictly associative unital".

The T-category $C'$ has objects which are continuous functors $E : C \to C$ together with continuous natural transformations

$$(E?)\oplus ? \xrightarrow{\cong} E(?\oplus ?).$$

Morphisms are continuous natural isomorphisms $E \xrightarrow{\cong} E'$ together with a commutative diagram

$$\begin{tikzcd}
(E?)\oplus ? \arrow[r, "\cong"] \arrow[d, "\cong"'] & E(?\oplus ?) \arrow[d, "\cong"] \\
(E'?)\oplus ? \arrow[r, "\cong"'] & E'(?\oplus ?).
\end{tikzcd}$$

The functor $\Gamma : C \to C'$ is (on objects)

$$X \mapsto X\oplus ?.$$

To prove that $C'$ is a T-category and that $\Gamma$ is continuous, the key point is to ge a more explicit description of the objects and morphisms of $C'$.

We see that an object $E \in Obj(C')$ is determined by the object $X = E(0)$ and a continuous choice of isomorphisms

$$X \oplus Y \xrightarrow{\cong} Z,$$

$$Y \in Obj(C).$$

Therefore,

$$Obj(C') = Obj(C) \times_{Map(Obj(C),Obj(C))} Map(Obj(C), Mor(C))$$

where

$$Obj(C) \to Map(Obj(C), Obj(C))$$

is the adjoint to $\oplus$, and

$$(44) \qquad Map(Obj(C), Mor(C)) \to Map(Obj(C), Obj(C))$$

is $Map(Id, S)$. Similarly, morphisms $E \to E'$ are determined by $E, E'$ and a morphism

$$E(0) \xrightarrow{\cong} E'(0).$$

Therefore,

$$Mor(C') = Mor(C) \times_{Map(Obj(C),Obj(C))^2} Map(Obj(C), Mor(C))^2$$

where the Cartesian coordinates of

$$Mor(C) \to Map(Obj(C), Obj(C))^2$$

are adjoint to $S?\oplus?$, $T?\oplus?$. The key point of proving that $C'$ is a (permutative) T-category is that (44) is a fibration, and hence so are $S_{C'}$, $T_{C'}$.

The continuous inverse of the functor $\Gamma$ is

$$E \mapsto E(0).$$

$\square$

9.4. **Singular vectors in Verma modules.** Consider the Verma module $V_{p,q} = V(c_{p,q}, 0)$ over the NS algebra (cf. [23]) where $c_{p,q}$ is given by (37). It is known ([29]) that $V_{p,q}$ has two singular vectors, one of which is

$$G_{-1/2}1$$

and the other, which we denote by $w$, has degree

$$\frac{1}{2}(p-1)(q-1).$$

In this subsection, we will compute some information about the singular vector $w$. Although [29] do compute a certain projection of $w$, their projection appears to annihilate the terms we need, and we were not able to find another reference which would include them.

Denote

$$V'_{p,q} = V_{p,q}/\mathcal{A} \cdot (G_{-1/2}1)$$

where $\mathcal{A}$ is the NS algebra. Then $V'_{p,q}$ has a basis consisting of vectors

$$G_{m_1}\dots G_{m_k}L_{n_1}\dots L_{n_\ell} \tag{45}$$

where

$$m_i\in\mathbb{Z}+\tfrac{1}{2}, n_j\in\mathbb{Z},$$

$$-\tfrac{3}{2}\geq m_1>m_2>\dots>m_k, -2\geq n_1\geq n_2\geq\dots\geq n_\ell.$$

Let, for $(\alpha_0,\alpha_1,\alpha_2,\dots)$, $\alpha_i\in\mathbb{N}_0$, $(\alpha_0,\alpha_1,\dots)>(\beta_0,\beta_1,...)$ if there exists an $i$ such that $\alpha_j=\beta_j$ for $j<i$ and $\alpha_i>\beta_i$. Let $I$ be the set of all sequences $(\alpha_0,\alpha_1,\dots)$ of non-negative integers where for all but finitely $i$, $\alpha_i=0$.

We introduce an $I$-indexed increasing filtration on $V_{p,q}$ where

$$F_{(\alpha_0,\alpha_1,\dots)}V'_{p,q}$$

is spanned by all elements (45) such that $(\beta_0,\beta_1,\dots)\leq(\alpha_0,\alpha_1,\dots)$ where $\beta_0=k+\ell$ and $\beta_i$ is the number of $m_i$ (resp. $n_j$) equal to $-1-(i/2)$.

**Lemma 15.** *The projection $w'\in V'_{p,q}$ of $w$ is, up to non-zero multiple, equal to*

$$G_{-5/2}G_{-3/2}L_{-2}\dots L_{-2}+\lambda L_{-2}\dots L_{-2}\ \textit{if } (p-1)(q-1)\textit{ is even} \tag{46}$$

$$G_{-3/2}L_{-2}\dots L_{-2}\ \textit{if } (p-1))(q-1)\textit{ is odd} \tag{47}$$

*plus elements of lower filtration degree, where $\lambda$ is an appropriate non-zero number.*

*Proof.* Consider the highest filtration monomial summand $q$ of the form (45) of $w'$. Assuming our statement is false, then the filtration degree of $q$ must be lower than the filtration degree of (46) (resp. (47)).

**Case 1:** $\beta_1=0$. Then let $i$ be the lowest such that $\beta_i>\alpha_i$ (this must exist by dimensional reasons). If $i-1$ is even, let

$$u=L_{(i-1)/2}w',$$

if $i-1$ is odd, let

$$u=G_{(i-1)/2}w'.$$

In either case, the highest filtration degree of $u$ is

$$(\beta_0,1,\beta_2,\dots,\beta_{i-1},\beta_i-1,\beta_{i+1},\dots,...).$$

In particular, it cannot cancel with $L_{(i-1)/2}$ resp. $G_{(i-1)/2}$ being applied to lower filtration summands of $w'$. This contradics $w'$ being a singular vector.

**Case 2:** $\beta_1 = 1$. Let $i$ be the lowest such that $\beta_i > \alpha_i$. If $i-2$ is even, let

$$u = L_{(i-1)/2}w',$$

if $i-2$ is odd, let

$$u = G_{(i-2)/2}w'.$$

The rest of the argument is the same as in Case 1. Now in the case of $(p-1)(q-1)$ even, note that

$$G_{1/2}(G_{-5/2}G_{-3/2}L_{-}2 \dots L_{-2})$$

produces a summand of

$$G_{-3/2}L_{-2} \dots L_{-2},$$

which can cancel only with

$$G_{1/2}(L_{-2} \dots L_{-2}).$$

$\square$

**Theorem 16.** *The quotient of $L_{p,q}$ by the sum of images of $a$ where $a$ are the coefficients of $Y(u,z)$ at $z^{\geq 1}$ with $u \in L_{p,q}$ is generated by*

$$(L_{-2})^i, G_{-3/2}(L_{-2})^i \tag{48}$$

*with $0 \leq i < (p-1)(q-1)/4$ if $p,q$ are odd, and $0 \leq i < ((p-1)(q-1)+1)/4$ if $p,q$ are even (and $p - q \equiv 2 \mod 4$). In particular, $L_{p,q}$ satisfies the $C_2$ condition (cf.* [14]*).*

*Proof.* Similar to [14]. Modulo lower filtration degrees, all elements (45) are in the submodule $C_2L_{p,q}$ generated by $(coeff_{z^{\geq 1}}Y(?,z))L_{p,q}$ unless $n_\ell = -2$ (or $\ell = 0$) and either $k = 0$, or $k = 1$ and $m_1 = -3/2$, or $k = 2$ and $m_2 = -5/2$. By Lemma 15, and Lemma 3.8 of [14], then, the listed elements generate the quotient $L_{p,q}/C_2L_{p,q}$. $\square$

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