Modular functors and their K-theory realisation

Joint work with Lehang Lai

Rational conformal field theory

Moduli space of Riemann surfaces with parametrized boundary

Gluing
Modular function: set $\Lambda$ of labels (finite)

\[ \mathcal{M}_\Lambda = \left\{ \Sigma, f : \text{component of } \Sigma \to \Lambda \right\} \]

Holomorphic f.d. vector bundle on $\mathcal{M}_\Lambda$ "compatible, under gluing"

\[ \Sigma \quad \longrightarrow \quad \mathcal{M}_\Sigma \]

surface with labelled
$2$ components

$$\Sigma_1 \cup \Sigma_2 \Rightarrow \Sigma_1 \otimes \Sigma_2 \quad \text{coherence}$$

$$M^\Sigma \cong \bigoplus_{\Delta \in \Lambda} M_{\Sigma, \Delta}$$

no additional component have same label.

A typical example: WZW model "fusion rules of level $k$ representations of $G$ is simply connected simple Lie group"
Desderatman: A realisation in $K$-theory

to simplify, suppose the bundle on the moduli space is trivialised. $\Rightarrow$ 2-dim. TQFT valued in vector spaces

$\mathbb{C} \oplus \mathbb{C}$

the category of f.d. \( \mathbb{C} \)-vector spaces, $\oplus, \otimes$

$K$-theory is a rigid commutative ring on $\mathbb{C}$-modules (stable homotopy theory).
generalised cohomology theories...

⇒ 2-dim TQFT valued in $K$-modules

\[ \bigvee K = M \]

\[ \wedge \]

In our paper, we made all this rigorous and constructed the realisation.
In the case of holomorphic modular forms, the semi-group of automorphic moduli space

\[ \mathbb{C} \mathbb{C} \mathbb{Q} \]

holomorphic central extension (classified by a single complex number, called central class).

En passant, we twist $K$-modules parametrized over $K(\mathbb{Z}, 3)$. 
More on the context:

Example: The Quillen determinant $\Lambda = \text{tr}^+$

$F: \text{Hol}(\Sigma) \xrightarrow{\otimes^0} \Omega^0(T\Sigma)$

Fredholm operator

$\text{Det}(F) = \text{Det}(\Sigma)$

Religne $\sim 1985$: To satisfy gluing, $\text{Det}(\Sigma)$ is a

[Further explanation or expression]
A hyper-kernel has dimension 1, or 1.

An example which is even worse: moduli space of Riemann surfaces with group structure.

Chiral fermion modular functor

\[ \text{ch} (\mathcal{H} \ominus \mathcal{E}) \rightarrow \Omega_+ (\mathcal{E}) \]
well defined for antiperiodic

\[ \gamma \text{ is normal} \]

periodic \rightarrow 2\text{-category due to Deligne:} \]

\[ R \]

\[ \text{Objects: } \mathcal{C} - \text{Clifford algebras} \]

\[ \text{1. Morphisms: Monic equivalences} \]

\[ \text{2. Morphisms: } \sim \text{ in modules} \]

\[ \sum K \]

Fermionic conformal blocks.
How does one get examples?

One route: Modular tensor categories (Dunkl, Bakalov - Kirillov)

→ all the data of a modular functor except the holomorphic part (c mod some number)

do not handle "fermionic" cases

K-theory realizations
Huang: rigorous constructions of modular tensor categories from vertex algebras.

describe additional structure on modular tensor categories, which give a fermion-like structure.

\[ \Theta, \Xi \subseteq \sigma_{VW} : V \otimes W \simeq W \otimes V \quad \text{(do not require)} \]
\[ \sigma_{VW} \sigma_{WV} = \text{Id} ! \]

- all objects strongly dualizable.
- balance: \[ \Theta_V : V \rightarrow V \]

\[ \Theta_{VW} = \sigma_{VW} \sigma_{WV} (\Theta_V \Theta_W) \]
\[ \Theta_{1} = \text{Id} \]
\[ \Theta_{V^*} = (\Theta_V)^* \]
< modular, semisimple, finitely many simple objects

\[ \Lambda = \{ V_i \} \]

\[ s_{ij} = \theta_i \theta_j^{-1} + \theta_i \theta_j \]

non-normal matrix

Bakalov - Kirillov book

\[ \Rightarrow \text{topological part of modular } \Rightarrow \text{twisted } \]

K-theory realization

Example: Freed - Hopkins - Teleman:

\[ K_{\ell,i}^*(G) = \text{Verlinde algebra of the} \]
1. Modular functor of WZW model on $G$.

$Z[\Lambda]$, fusion rules $\tau_i$.

$d_i, d_i' = \sum_\gamma \tau_i \gamma \gamma^{-1}$.

Using Huang, Bakalov - Kirillov, Frenkel, K. L.

Modular functor $\Rightarrow K_{G/2}(G) \rightarrow K(2,3)$.

Framed structure: $V \quad 2 \cdot V \otimes V \cong 1$

$\Theta_{V^*} = -1$. 

WZW
Example: N = 1 - supersymmetric minimal models.

representation of (super-)semi-group of super-algebra