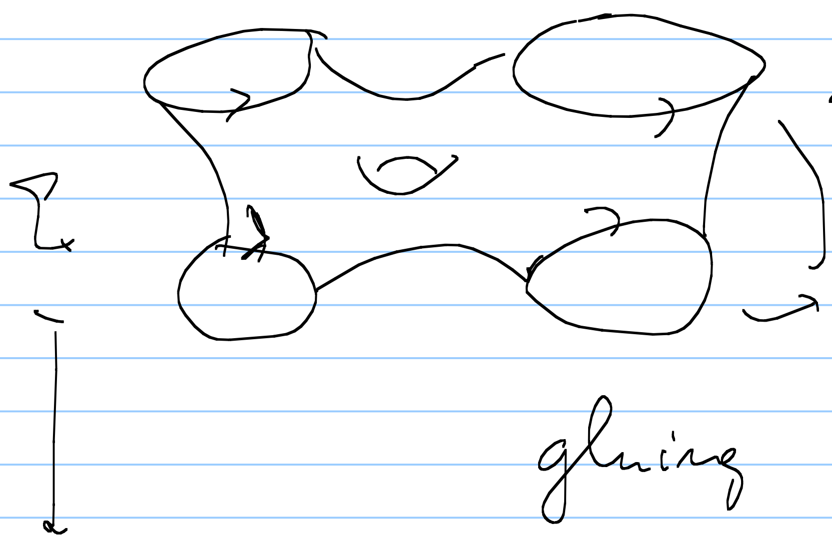
Note Title 11/7/2014

Modular functors and their K-theory realization

Joint work with Uchang Lai

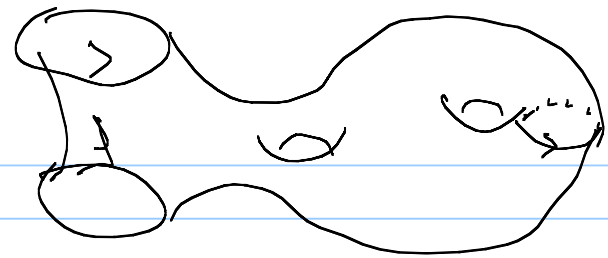
Rational conformal field theory



Moduli space of Riemann surfaces with parametrised boundary
complex structure

gluing

Σ



Modular functor: – set Λ of labels (finite)

(Segal) $\mathcal{M}_\Lambda = \{ \Sigma, f : \{\partial \text{ components of } \Sigma\} \to \Lambda \}$

Holomorphic f.d. vector bundle on $\mathcal{M}_\Lambda$ "compatible under gluing"

$$\Sigma \longmapsto M_\Sigma$$

surface with Λ-labelled ∂

∂ components

$$M_{\Sigma_1 \sqcup \Sigma_2} \cong M_{\Sigma_1} \otimes M_{\Sigma_2}$$

$$M_{\Sigma}^{\vee} \cong \bigoplus_{\lambda \in \Lambda} M_{\Sigma, \lambda}$$

} coherence diagrams.

↑ two additional component have same label λ

A typical example : WZW model (chiral)

("projective version")

"fusion rules of level k representations of $\widetilde{LG}$, G simply connected simple Lie group"

Desideratum : A realisation in K-theory

to simplify, suppose the bundle on the moduli space is trivialized. $\Rightarrow$ 2-dim. TQFT valued in vector spaces.

$\bigoplus \mathbb{C}_2$

the category of f.d. $\mathbb{C}$-vector spaces, $\oplus, \otimes$

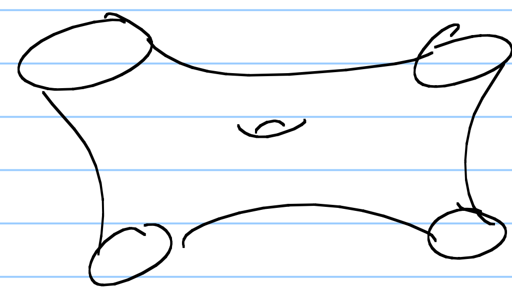
K-theory is a rigid commutative ring in S-modules (stable homotopy theory).

generalised cohomology theories ....

$\Rightarrow$ 2-dim TQFT valued in K-modules

$\bigvee K = M$
$\wedge$

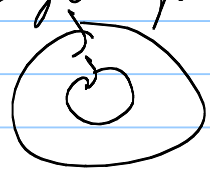
$\Lambda_K M_{in} \to \Lambda_K M_{out}$



In our paper, we made all this rigorous and constructed the realisation.

In the case of holomorphic modular functors,

semigroup of annuli ⊂ moduli space



↗

holomorphic central extension (classified by a single complex number called central charge)

c

$$\boxed{c \in \mathbb{Q}}$$

EM space

⇝ twisted K-modules parametrized over $\widetilde{K(\mathbb{Z},3)}$.

More on the context:

Example: The Quillen determinant $\Lambda = \{1\}$

$$F : Hol(\Sigma) \longrightarrow \Omega^0_+(\partial\Sigma)$$

(with $Hol(\Sigma) \to \Omega^0(S^1) \to \Omega^0_+(\partial\Sigma)$)

F: Fredholm operator

$Hol(\Sigma)$: holomorphic functions

$\Omega^0_+(\partial\Sigma) = \bigoplus_{\partial} \Omega^0_+(S^1)$

$\Omega^0_+(S^1)$ = functions on S^1 which extend holomorphically to D.

$$Det(F) = Det(\Sigma).$$

Deligne ~ 1985: To satisfy gluing, $Det(\Sigma)$ is a super-line.

↑ has dimension 1, or −1.

Atiyah-Singer — Self-adjoint (skew) Fredholm operators.

An example which is even worse: moduli space of Riemann surfaces with spin structure.

<u>Chiral fermion modular functor</u>

(Pressley-Segal)

$$\mathrm{Pf}\left(\Omega^{1/2}(\Sigma) \longrightarrow \Omega^{1/2}_{+}(\partial\Sigma)\right)$$

↗

K. 2003
Annales Sci. ENS

well defined for antiperiodic
∂ components

NS

sym. monoidal
periodic → 2-category due to Deligne: $\mathbb{C}$

R

Objects : $\mathbb{C}$ - Clifford algebras
1-Morphisms : Morita equivalences
2-Morphisms : $\cong$.

$\mathbb{C}\{z\}$ $z^2 = 1$

bimodules

Σ K.

Fermionic conformal blocks.

How does one get examples?

One route: Modular tensor categories (Drinfeld, Bakalov-Kirillov)

$\Downarrow$

all the data of a modular functor except the holomorphic part (c mod some number)

$\Downarrow$

K-theory realizations

↗ do not handle "fermionic" cases

Huang: rigorous constructions of modular tensor categories from vertex algebras.

describe additional structure on modular tensor categories which give a fermionic structure.

$\oplus$, $\boxtimes$ $\leftarrow$ $\sigma_{VW} : V \boxtimes W \xrightarrow{\simeq} W \boxtimes V$ (do not require $\sigma_{VW}\sigma_{WV} = Id$!)

all objects strongly dualizable.

- balancing $\theta_V : V \to V$

$\theta_{V \boxtimes W} = \sigma_{WV}\sigma_{VW}(\theta_V \boxtimes \theta_W)$

$\theta_1 = Id$

$\theta_{V^*} = (\theta_V)^*$

- modular, semisimple, finitely many simple object

$(\Lambda =)$ $\{V_i\}$ $s_{ij} = \theta_i^{-1} \theta_j^{-1} \operatorname{tr}(\theta_{V_i^* \boxtimes V_j})$

$\nearrow$

non-singular matrix.

Bakalov - Kirillov book.

$\Rightarrow$ topological part of modular $\Rightarrow$ twisted K-theory realisation.

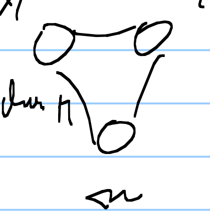
Example: Freed - Hopkins - Teleman:

$K^*_{G,\tau}(G) \cong$ "Verlinde algebra of the

↗ modular functor of WZW model on G"

$\mathbb{Z}[\Lambda]$, fusion rules

$$\lambda_1 \cdot \lambda_2 = \sum_{\mu} \dim H \; \cdot \mu$$



Using Huang, Bakalov–Kirillov, Drinfeld, K.L.

Modular functor WZW $\Longrightarrow K_{G,\tau}(G) \longrightarrow K(\mathcal{C},3)$.

Fermionic structure on a modular: V^- $\quad z : V^- \boxtimes V^- \cong 1$, $\theta_{V^-} = -1$ }, tensor category $\sigma_{V^- V^-} = -1$)

Example: $N=1$ - supersymmetric minimal models.

representations of (super)-semigroup of super-annulo.