Equivariant (and non-equivariant)

Homotopy theory I

Solution of the Kervaire invariant I problem
Hill, Hopkins, Ravenel

What is homotopy theory?

Spaces $X$, $f: X \to Y$ continuous

Homotopy $X \times I \to Y$

Identify maps up to homotopy — analysis (e.g., totally disconnected spaces)
Weak equivalence

\[ \pi_n(X) = [S^n, X] \text{ homotopy classes} \]

homotopy groups

\[ X \text{ space } \ast \in X \]

A weak equivalence is a map \( f : X \rightarrow Y \)

which induces a bijection on \( \pi_0 \) (set of path-connected components)

and for every \( x \in X \), an isomorphism

\[ \pi_n(X, x) \rightarrow \pi_n(Y, f(x)) \quad \forall \ x \in X. \]

The derived category of spaces: Take the category of spaces (and continuous maps), invert weak equivalences.

\[ \text{Spaces} \xrightarrow{\text{universal among functors where weak equivalence go to isomorphisms}} \text{D Spaces} \]
**Stable homotopy theory**

\[ \Sigma X = X \times I / (x, 1) \sim (y, 1), \quad (x, 0) \sim (y, 0) \]

**Homology:**

1. \( H_n X \equiv H_{n+1} \Sigma X \) \{ stable \}

**What is homology?**

- **Axiom 1 & Axiom 2**
- **Functors**
  \( H_n : \text{spaces} \rightarrow \text{Ab. groups} \)

2. For a nice embedding \( X \subseteq Y \), an exact sequence

  \[ \text{Eilenberg - Steenrod Axioms} \]
\[ H_m X \rightarrow \tilde{H}_m Y \rightarrow \tilde{H}_m Y/X \]

Folows:
\[ H_{m-1} X \rightarrow \tilde{H}_{m-1} Y \rightarrow \tilde{H}_{m-1} Y/X \]

Technical axioms to deal with infinity.

Dimension axiom: \[ \tilde{H}_m S^0 = H_m \ast = 0 \text{ for } m \geq 0 \]
\[ H_m X = \tilde{H}_m (X \cup \ast \ast) \]
\[ A \text{ for } m = 0. \]

Abelian group.

Singular homology theory is characterised by the above axioms.

Omitting the dimension axiom \[ \Rightarrow \text{ generalised homology} \]
Cohomology - defined similarly as homology

contravariant functor

\( H^n : \text{spaces} \rightarrow \text{abelian groups} \)

\[ H^n X \leftarrow H^n Y \leftarrow H^n Y/X \text{ exact} \]

[generalised cohomology - examples to come (K-theory, bordism)]

Next step: build a stable derived category

suspension \( \Sigma \) = equivalence of categories

Adams, May, ....
May: spectra = sequence of spaces $\langle E_n \rangle$

\[ E_n \xrightarrow{f_n} E_{n+1} \quad n \in \mathbb{N} \]

\[ X \xrightarrow{\phi} \text{space of based maps, } \Sigma \xrightarrow{\psi} X \]

\[ \text{Norphism } (\langle E_n \rangle) \xrightarrow{f} \langle T_n \rangle : \text{ continuous map} \]

\[ E_n \xrightarrow{f_n} T_n \]

\[ \Sigma \xrightarrow{\phi \circ f_n \circ \iota} \Sigma T_{n+1} \]

\[ \Sigma \xrightarrow{f_{n+1}} \]
A weak equivalence of spectra is a morphism $f: E \to F$ which induces an isomorphism on $\pi_n E \to \pi_n F$ for $n \in \mathbb{Z}$.

$S^m = \bigcup_{n \in \mathbb{Z}} \Sigma^n \Sigma^{\infty} X$

$\Sigma^n X \cong \lim_{\rightarrow} X^{k+n}$
The stable derived category: category of spectra with equivalences inverted.

\[ \text{Spectra} \rightarrow \text{D Spectra} \]

universal among functors in which equivalence go to monomorphims.

Ex: D Spectra contains the derived category of chain complexes (of abelian groups).

\[ A \rightarrow 0 \rightarrow A \rightarrow 0 \cdots \]

D Chain $\cup_{n \geq 0} \text{dim}_0 \Rightarrow \text{Ext}^n_{\mathbb{Z}}(A, B)$

(non-trivial for $n = 0, 1$).
D chain \rightarrow D Spectrum

A \rightarrow HA Eilenberg - Mac Lane spectrum.

represents singular homology 
cohomology.

In general, D Spectrum represent generalized homology and cohomology. If $E,F$ are spectra,

$E^m F = [E, \Sigma^m F]$ generalizes $Ext$

$E_n F = \pi_n (E \wedge F)$ (can analogue of $\otimes$ of chain complexes)
For a space $X$ with base point,

$$\tilde{E}^n X = \text{def} \quad \tilde{E}^n \Sigma^\infty X \quad \tilde{E}_n X = \text{def} \quad E_n \Sigma^\infty X.$$

Generalized homologies and cohomologies come in pairs, and correspond to spectra (more precisely, to objects of a spectrum).

Example 1: singular homology and cohomology with $\mathbb{Z}$-coefficients. In $A$ we have $HA = (\mathbb{Z}_n)$

$$\mathbb{Z}_n = K(A, n) \in n\text{-th homotopy group} = A$$

$$= \ast \quad \text{else}, \quad n > 0$$
$K^0 X = KX$ extends to a generalised bordism theory

universal admissible group on a commutative monoid $\mathbb{G}$

$\text{Aut}(\mathbb{G}) = \text{Aut}(\mathbb{G})$

Example (Atiyah – using $B\mathbb{G}$)

$X = \text{polyhedral cone}$

family of vector spaces

continuously parametrised monoid $\mathbb{G}$
\[ \{ \text{classes of complex vector bundles of dim. m} \} = [X, BU(m)] \]

\[ f^* \xi \rightarrow f^* \eta \]

\[ f : X \rightarrow BU(m) \]

\[ m \text{-dim. complex vector bundle } \xi \]

\[ \text{(universal bundle)} \]

\[ BU(m) \xrightarrow{S} AU(m+1) \quad \lim_{m \to \infty} BU(m) = BU \]

\[ \text{vector bundle } \xi \rightarrow \xi \oplus 1 \]

**Theorem (Bott):** \[ \Omega^2(BU \times \mathbb{Z}) \cong BU \times \mathbb{Z} \]

\[ K^{2n}X = [X, BU \times \mathbb{Z}] \]
\[ X^{k+n+1} = \left[ X, \mathbb{S} \left( \mathbb{R}^k \times \mathbb{R} \right) \right] \]

\[ U = \lim_{m \to \infty} U(m). \]

Example: Cobordism (Pontryagin-Thom)

n-manifolds (compact smooth) trivial bundle

Stable normal data: \( V \oplus \Sigma_n \cong N \)

\( \mathbb{R}^N \leftarrow \mathbb{R}^N \)

structure

- oriented
- complex
- trivial (framed)
- no structure (unoriented)

Two manifolds are cobordant
\[ \pi_{1+n} \rightarrow \text{ manifold } V \text{ with the same kind of stable normal data whose boundary is } N_1 \cup N_2 \]

\[ \text{(set of bordism classes of } n \text{- manifold, } \Omega \text{)} \]

\[ \text{any flavor} \]

= abelian group \(\pi_{1+n} \)

oriented \(\Omega D_n\)
unoriented \(\Omega O_n\)
complex \(\Omega U_n\)
framed \(\Omega \times_n\)
many we can construct a Thom spectrum $\mathbb{M}U$
such that $\mathbb{M}U_n = \mathbb{P}_n \mathbb{M}U$ etc.

$\pi_0, \pi_{\infty}, \mathbb{M}$

let's just do $\mathbb{M}U$.

Start with $BU(n)$, universal bundle $\gamma^n$.

The Thom space: $BU(n)^{\gamma^n} = \lim_{X \subset BU(n) \text{ compact}} 1 \text{-point comp. of total space} \gamma^n \times X$

$\Sigma^2 BU(n)^{\gamma^n} \rightarrow BU(n)^{\gamma^{n+1}}$
\[ \Omega U(n) \subset B U(n+1) \quad \prod_k \Omega U = \Omega \left[ x_1, x_2, \ldots \right] \]
\[ \delta^n \times \mathbb{C} \rightarrow \delta^{n+1} \quad \dim x_i = 2i \quad \text{(Quillen)} \]

\[ \Omega U = (\mathbb{C}) \quad \mathbb{Z}_k = \lim_{\longrightarrow} \mathbb{Z}^{k+2n} \]

For trivial bundles, the classifying space is formed.

\[ \mathcal{M}^* = S\Sigma S^0 = S \]

The \( S = \lim_{\longrightarrow} \mathbb{Z}^{k+2n} \) is the \( k \)-th homotopy group of spheres.
Some details:

\[ \mathcal{B}u(n) = \{ \text{n-dimensional subspaces of } \mathbb{C}^n \} \]

\[ \mathbb{C}^\infty = \bigcup_n \mathbb{C}^n \]

Basic idea of the cobordism group identification:

1. Manifold \( M \) \( \rightarrow \) homotopy group of Thom space
Pontryagin - Thom construction (say, complex case)

\[ M \subset S^N \]

\( m \)-manifold

Tubular neighborhood \( \cong \) total space of normal bundle

classifying map \( E \mathbb{V}^m \)

\[ U \rightarrow \text{total space of } \mathbb{V}^m \text{ on } BU(n) \]

\[ M \cup U \rightarrow * (= BU^{\mathbb{V}^m} \backslash E \mathbb{V}^m) \]

(2) homotopy groups of Thom spaces \( \rightarrow \) manifolds
complex case:
\[ \alpha \in \prod_{n + 2k} \text{BU}(k)^{n+k} \]

\[ \rightarrow \text{n-manifold with complex stable normal bundle} \]

Transversality (A)

in the fiber direction

Approximate \( \alpha \) by a smooth map transverse

to the 0-section of \( \text{BU}(k)^{n+k} \)

\[ M = \alpha^{-1}(0\text{-section}) \]

Normal bundle complex (induced from \( \gamma_k \))