Equivariant and non-equivariant

homotopy theory II

Equivariant homotopy theory
- group $G$ acting $G = \text{finite group}$ or compact Lie group
- object = spaces with $G$-action
- morphisms = maps preserving $G$-action
- $G$-equivalence $f: X \to Y$ which induces a (non-equivariant) weak equivalence
\[ f^H : X^H \to Y^H \quad \text{for every} \quad (\text{closed}) \quad H \leq G. \]

Derived category of \( G \)-spaces

\[ G \text{-Spaces} \longrightarrow \mathsf{D} G \text{-spaces} \]

a universal functor among functors which take \( G \)-equivaleces into isomorphisms.

Stable story \( - \) suspensions with respect to representations (f.d. real \( G \)-representations \( V \))

\[ \Sigma^V X = X \wedge S^V \]
\((X \wedge Y = X \times Y / (\ast \times Y) \cup (X \times \ast))\)

Spaces with base points

\(S = 1\)-point compactification of \(V\)

We want to build a derived category on which \(\Sigma^n\) is a (self)-equivalence of categories.

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May: Consider a sequence of irreducible \(G\)-representations \(V_0, V_1, V_2, \ldots\) (LNM 1213)
such that every irreducible occurs infinitely many times.

\[ Z V_0 + \cdots + V_m \to \Omega V_{m+1} \]

Maps of \( \infty \)-preets \( f:(Z V) \to (T V) \) are collections of \( \infty \)-equivariant maps

\[ f_v : Z_v \to T_v \]

commuting with structure maps.

\( \infty \)-equivalences of \( \infty \)-preets : maps

\[ f : (Z V) \to (T V) \]
of $\text{G}$-spectra such that each $f_v$ is a $G$-equivalence.

**Derived category of $G$-spectra**

$G$-spectra $\rightarrow$ $G$-spectra

universal among all functors which take $G$-equivalences into isomorphisms

**Suspension spectrum** $X$ is given

$\Sigma^\infty X = (\mathbb{Z}_v)$

$\mathbb{Z}_v : \text{colim}_w \bigwedge^{v+w} \mathbb{Z}_w X$
Sphere spectrum

\[ S = S^0 S^0 \]

\[ V, W \text{ irreducible } \Lambda^* \text{- representations} \]

\[ S^{V-W} = S[V][W] \]

Only depends on \[ V-W \in RO(\mathcal{G}) \]

\[ \uparrow \]

The real representation

\[ \cong \text{class of } G \]

\[ = K \left( \text{f.d. real } \mathcal{G}^* \text{- representations } \Theta \right) \]

\[ = 2 \text{d irreducible } \Lambda^* \text{- representations} \]
Example: \( G = \mathbb{Z}/2 \) \( R_0(\mathbb{Z}/2) = \{1, \alpha\} \)

IR fixed \( IR \),

So I have \( S^{k+l\alpha} \), \( k, l \in \mathbb{Z} \)

For \( G \)-spectra \( E, F \), \( V \in R_0(G) \), define \( E^V F = [E, \Sigma^V F] \) up to morphisms in \( \mathcal{D} \)-spectra

\( E_V F = [S^V, E \wedge F] \) symmetric monoidal structure (technical)
$X$ $G$ - space

$\tilde{E}^n X = E^n \mathcal{E}^\infty X$, $\tilde{E}^n X = E^n \mathcal{E}^\infty X$.

$V \in \text{RO}(G)$ \{ RO(G)-graded (generalised) $G$-equivariant (co)homology \}

Examples: What is the analogue of
the $D$-chain complex $< D$-spectra

$G$-equivariantly? \\
Answer: $D$-chain complexes of $G$-Nakayama functors
Equivariant $K$-theory - constructed similarly to $K$-theory, $G$-equivariant bundles.

Bott periodicity not elementary -

Atiyah-Singer index theorem

Equivariant cobordism $MU$


- does not compute "naive" $G$-cobordism groups.
\[ S^{n+2N} \xrightarrow{f} BU(N)^{\mathbb{R}} \xrightarrow{w} (\text{complex non-equiv. case}) \]

Transverse to \( BU(N) \)

to 0- section \( O \)- section

\( f^{-1}(0 \text{-section}) \) is the manifold.

This fails equivariantly.

Another set of examples \( (c = \mathbb{R}/2) \)

Real homotopy theory

Example: Atiyah’s Real \( K \)-theory
A real bundle on a \( \mathbb{Z}/2 \)-space \( X \) is a complex bundle with an antilinear involution. (If \( X \) is fixed, this is an equivalent category to real vector bundles on \( X \) - via taking \( \mathbb{Z}/2 \)-fixed points.)

Non-equivariant real \( K \)-theory:

\[ K_0(X) \] is 8-periodic
$m \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$

$\pi_n KO \quad \pi_0 \pi_1 \pi_2 \pi_3 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$

$\pi_n K \quad \pi_0 \quad \pi_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$

Atiyah Real K-theory (RO($\mathbb{Z}_2$) - graded equivariant generalized cohomology theory)

$KIR^{k + \ell \alpha}(X)$

is $(1 + \alpha)$ - periodic! K-theory and reality

Proof is elementary (Atiyah).
Real cobordism  

\[ \Sigma \text{BU}(n) \xrightarrow{\text{H}} \text{BU}(n+1) \]

\( \mathbb{Z}/2 \) acts by complex conjugation.

Hill–Hopkins–Ravenel: \( \mathbb{Z}/2 \) - action

\[ \text{MU} \wedge \text{MU} \wedge \text{MU} \wedge \text{MU} \]

Complex conjugation detects the Kervaire invariant 1 elements.
Techniques for calculations

Let $G$ be a finite group and consider a mapping $x \to y$.

There exists an exact sequence:

$$E_G \longrightarrow S \longrightarrow \tilde{E}_G \longrightarrow \text{cone}$$

in homology with distinct base point $y$.

$E_b$ = "nice" $b$-space

$E_G \overset{\sim}{\longrightarrow} \ast$ non-equivariantly

$E_G^H = \emptyset$ if $H \leq G$
$g = 2/2$: There is a model for $\widetilde{E}^g$:

$$S^a \leq S^{2a} \leq S^{3a} \leq \ldots \leq S^{\infty a} \mid \widetilde{E}^g.$$ 

The Tate diagram for a $\mathcal{O}$-spectrum $E$

\begin{equation*}
\begin{align*}
E_{G_T} \wedge \mathcal{O} & \to \mathcal{O} \to \mathcal{E}_G \wedge \mathcal{O} \\
\downarrow \sim & \downarrow \downarrow \\
E_{G_T} \wedge \mathcal{F}(E_{G_T}, \mathcal{O}) & \to \mathcal{F}(E_{G_T}, \mathcal{O}) \to \mathcal{E}_G \wedge \mathcal{F}(E_{G_T}, \mathcal{O}) \\
\downarrow \sim & \downarrow \downarrow \\
(\mathcal{O}) \wedge \mathcal{F}(E_{G_T}, \mathcal{O}) & (\mathcal{O}) \wedge \mathcal{F}(E_{G_T}, \mathcal{O})
\end{align*}
\end{equation*}
$E = (\mathbb{Z}_2^\infty)$

$E^c = (\mathbb{Z}_m^c)$

non-equivariant spectrum

$\Pi_m E = \Pi_m E^c$

$_{m \in \mathbb{Z}}$

Adams isomorphism

geometric fixed points $\Phi E$

$E G^+ \wedge_G E \rightarrow E^c \rightarrow (E G \wedge E)^c$

$\approx$

$E G^+ \wedge_G E \rightarrow \text{spectral sequence}$
Example: \( \Phi^2_{\text{tr}}(\mathbb{H} \mathbb{R}) = \mathbb{M} \mathbb{O} \)

\[ \Phi^2_{\text{tr}} K_{\mathbb{H} \mathbb{R}} = \ast \]

1) \( \Phi^2_E \rightarrow E \) is an equivalence,

\( E \rightarrow F(E_{2/4}, E) \) is an equivalence

(a complete spectrum)
What is a spectral sequence (homological)

abelian groups

$E^r_{p,q}$, \( p,q \in \mathbb{Z}, \ r = 1, 2, 3, \ldots \)

$d^r : E^r_{p,q} \rightarrow E^r_{p-r, q+r}$

$d^r d^r = 0$.

$E^{r+1}_{*,*} = H(E^r_{*,*}, d^r)$

If, for each \( p,q \), \( d^r \) eventually 0

on \( E^r_{p,q} \), then \( E^\infty_{p,q} = \lim_{r \rightarrow \infty} E^r_{p,q} \).
If I have a filtration on a space $X$, then $X \subseteq F_0 X \subseteq F_1 X \subseteq F_r X \subseteq \ldots$.

I need to define a spectral sequence in the sense of

$$E_{p,q}^1 = E_{p+2}^r (F_r X / F_{r-1} X)$$

and

$$E_{p+2}^\infty = E_p^\infty (E_{p+2} X)$$

and

$$F^b / \mathcal{P}^{b-1}$$
\[ K_R^* = \mathbb{Z} \{ \beta, \beta^{-1} \} \sigma^4, \sigma^{-4} \} \{ a \} / (2a, a^3) \]

\[ \text{dim.} 1+\alpha \]
\[ \text{Boett element} \]
\[ 4(1-\alpha) \sim \alpha \]
\[ \int_0^\infty f^\alpha \]

Recall
\[ M_R \times = \mathbb{Z} \{ x_1, x_2, \ldots \} \]
\[ |x_i| = 2i \]

Spectral sequence for \( M_R^* \):

Both \( K_R \) and \( N_R \) are complete.
\[ E^1 = \mathbb{Z} \{ x_1, x_2, \ldots \} \{ \sigma, \sigma^{-1} \} \{ a \} \]
\[ \dim (x_i) = i (1 + \alpha) \]
\[ \begin{pmatrix} 1 - \alpha & -\alpha \\ \end{pmatrix} \]
\[ d^{2^{n+1} - 1} (c^{2^m}) = \nu_a a^{2^{n+1} - 1} \]
\[ \nu_0 = 2 \quad \nu_i = x_i^{2^m - 1} \]
\[ \overline{E^\infty} = MIR_A^* (\text{Hau-K.}) \]