Note Title 1/16/2012

Equivariant and non-equivariant homotopy theory II

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Equivariant homotopy theory

- group G acting $\quad G$ = finite group or compact Lie group

Objects = spaces with G-action

morphisms = maps preserving G-action

<u>G-equivalence</u> $f: X \to Y$ which induces a (non-equivariant) weak equivalence

$f^H : X^H \to Y^H$ for every (closed) $H \subseteq G$.

Derived category of G-spaces

$$G\text{-Spaces} \longrightarrow D\,G\text{-spaces}$$

a universal functor among functors which take G-equivalences into isomorphisms.

Stable story - suspensions with respect to representations (f.d. real G-representations V)

$$\Sigma^V X = X \wedge S^V$$

$$\left(X \wedge Y = X \times Y / (* \times Y) \cup (X \times *) \right)$$

spaces with base points

S^V = 1-point compactification of V

We want to build a derived category on which Σ_1^V is a (self)-equivalence of categories.

<u>May</u>: Consider a sequence of (f.d.) irreducible G-representations $V_0, V_1, V_2, \ldots$ (LNM 1213)

such that every irreducible occurs infinitely many times.

$$Z_{V_0 + \cdots + V_n} \xrightarrow{\simeq} \Omega^{V_{n+1}} Z_{V_0 + \cdots + V_{n+1}}.$$

Maps of G-spectra $f : (Z_V) \to (T_V)$ are collections of G-equivariant maps

$$f_V : Z_V \to T_V$$

commuting with structure maps.

G-equivalences of G-spectra: maps

$$f : (Z_V) \to (T_V)$$

of G-spectra such that each f_V is a G-equivalence.

Derived category of G-spectra

G-spectra $\longrightarrow$ $\mathcal{D}$ G-spectra

universal among all functors which take G-equivalences into isomorphisms.

<u>Suspension spectrum</u>: X G-space

$$\Sigma^\infty X = (Z_V)$$

$$Z_V = \operatorname*{colim}_W \Omega^{V+W} \Sigma^W X$$

Sphere spectrum

$$S = \Sigma^\infty S^0$$

V, W irreducible G-representations

$$S^{V-W} = S[V][-W]$$

only depends $V - W \in RO(G)$

the real representation ring of G

$\cong$ classes of

$= K(\text{f.d. real } G\text{-representations}, \oplus)$

$= \mathbb{Z}\{\text{irreducible } G\text{-representations}\}$

Example: $G = \mathbb{Z}/2$ $\quad RO(\mathbb{Z}/2) = \mathbb{Z}\{1, \alpha\}$

$\mathbb{R}$ fixed $\quad$ $\mathbb{R}$, sign representation

So I have $S^{k+l\alpha}$, $k, l \in \mathbb{Z}$

For G-spectra E, F, $V \in RO(G)$, define

$$E^V F = [E, \Sigma^V F] \Leftarrow \text{morphisms in } \mathcal{D}\ G\text{-spectra}$$

symmetric monoidal structure (technical)

$$E_V F = [S^V, E \wedge F]$$

X G-space

$$\tilde{E}^V X = E^V \Sigma^\infty X \,, \quad \tilde{E}_V X = E_V \Sigma^\infty X .$$

$V \in RO(G)$ } $RO(G)$-graded (generalized) G-equivariant (co)homology.

Examples: What is the analogue of the $\mathbb{D}$-chain cxes $\subset$ $\mathbb{D}$-Spectra G-equivariantly?

Answer: $\mathbb{D}$-chain complexes of G-Mackey Functors

Equivariant K-theory - constructed similarly to K theory, G-equivariant bundles.
Bott periodicity not elementary -
Atiyah-Singer index theorem

Equivariant cobordism MU_G
- via Thom spaces of classifying spaces of universal G-n-bundles.
- does not compute "naive" G-cobordism groups

$$\begin{tikzcd} S^{n+2N} \arrow[r, "f"] & BU(N)^{\gamma_N} \\ & BU(N) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

transverse $\pitchfork$ to 0-section ↑ 0-section

(complex non-equiv. case)

f^{-1}(0-section) is the manifold.

This fails equivariantly.

<u>Another set of examples ($G = \mathbb{Z}/2$)</u>

Real homotopy theory

<u>Example</u>: Atiyah's Real K-theory

A Real bundle on a $\mathbb{Z}/2$-space X is a complex bundle with an <u>antilinear</u> involution. (If X is fixed, this is an equivalent category to real vector bundles on X - via taking $\mathbb{Z}/2$ fixed points.)

<u>non-equivariant real K-theory</u>:

$KO(X)$ - is 8-periodic

n	0	1	2	3	4	5	6	7
$\pi_n KO$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0
$\pi_n K$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Atiyah Real K-theory ($RO(\mathbb{Z}/2)$-graded equivariant generalised cohomology theory)

$\mathbb{Z}/2$-space

$$KR^{k+l\alpha}(X)$$

is $\boxed{(1+\alpha)\text{-periodic}}$!

K-theory and Reality

Proof is elementary (Atiyah).

Real cobordism $M\mathbb{R}$ - Landweber

$$\Sigma^{1+\alpha} \underbrace{BU(n)^{\gamma^n}}_{\mathbb{Z}/2 \text{ acts by complex conjugation}} \to BU(n+1)^{\gamma^{n+1}}$$

Hill - Hopkins - Ravenel: $\mathbb{Z}/8$ - action

$$MU \wedge MU \wedge MU \wedge MU$$

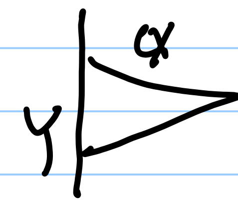
complex conjugation

detects the Kervaire invariant 1 elements.

Techniques for calculations

Let G be a finite group.

mapping cone $X \to Y$



$\Rightarrow$ Long exact sequence in homology

$$(*)\ \left\{ EG_+ \longrightarrow S^0 \longrightarrow \tilde{E}G \right.$$

disjoint base point

EG = "nice" G-space

$$\left.\begin{array}{l} EG \simeq * \quad \text{non-equivariantly} \\ EG^H = \emptyset \quad \{e\} \neq H \subseteq G \end{array}\right\} \text{G-free action}$$

$G = \mathbb{Z}/2$: There is a model for $\widetilde{EG}$:

$$\begin{array}{c} S^{\alpha} \subseteq S^{2\alpha} \subseteq S^{3\alpha} \subseteq \cdots \quad S^{\infty\alpha} \\ \hphantom{S^{\alpha} \subseteq S^{2\alpha} \subseteq S^{3\alpha} \subseteq \cdots \quad} \| \\ \hphantom{S^{\alpha} \subseteq S^{2\alpha} \subseteq S^{3\alpha} \subseteq \cdots \quad} \widetilde{EG}. \end{array}$$

<u>The Tate diagram</u> for a G-spectrum E

equivalence

$$\begin{tikzcd} EG_+ \wedge E \arrow[r] \arrow[d, "\sim"'] & E \arrow[r] \arrow[d] & \widetilde{EG} \wedge E \arrow[d] \\ EG_+ \wedge F(EG_+, E) \arrow[r] & F(EG_+, E) \arrow[r] & \widetilde{EG} \wedge F(EG_+, E) \end{tikzcd} \qquad \begin{array}{c} (*) \wedge E \\ \\ (T) \\ \\ (*) \wedge F(EG_+, E) \end{array}$$

$$E = (Z_V) \qquad\qquad E^G = (Z^G_n)$$

non-equivariant spectrum

$$\pi_n E = \pi_n E^G$$

$n \in \mathbb{Z}$

$G = \mathbb{Z}/2$

Adams isomorphism $\qquad$ geometric fixed points $\qquad \Phi^{\mathbb{Z}/2} E$

$$\begin{tikzcd}
✓\ EG_+ \wedge_G E \arrow[r] \arrow[d, "\sim"'] & E^G \arrow[r] \arrow[d] & (\widetilde{EG} \wedge E)^G\ ✓ \arrow[d] \\
✓\ EG_+ \wedge_G E \arrow[r] & ✓\ F(EG_+, E)^G \arrow[r] & ✓\ \hat{E}G
\end{tikzcd}$$

spectral sequences

$\uparrow$ Borel homology of E $\quad$ $\uparrow$ Borel cohomology of E $\quad$ $\uparrow$ Tate cohomology

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<u>Example</u>: $\Phi^{\mathbb{Z}/2}(M\mathbb{R}) = MO$

$\Phi^{\mathbb{Z}/2} K\mathbb{R} = *$

If $\Phi^{\mathbb{Z}/2} E \to \hat{E}$ is an equivalence,
$E \to F(E\mathbb{Z}/2_+, E)$ is an equivalence
(a <u>complete</u> spectrum)

What is a spectral sequence (homological)

abelian groups

$$E^r_{p,q} \qquad p,q \in \mathbb{Z} \quad , \quad r = (1), 2, 3, \ldots$$

$$d^r : E^r_{p,q} \longrightarrow E^r_{p-r,\,q+r-1} \qquad d^r d^r = 0.$$

$$E^{r+1}_{*,*} = H(E^r_{*,*}, d^r)$$

If, for each p,q, d^r eventually 0 on $E^r_{p,q}$ $\qquad E^\infty_{p,q} = \varinjlim E^r_{p,q}.$

If I have a filtration on a (spectrum) space X

$$\cdots \subseteq F_0 X \subseteq F_1 X \subseteq F_2 X \subseteq \cdots X$$

nice

union

$\exists$ spectral sequence (Serre)

$$E^1_{p,q} = E_{p+q}(F_p X / F_{p-1} X)$$

$$E^\infty_{p,q} = E^0_p(E_{p+q} X)$$

F^p/F^{p-1}

$$KR_\star = \mathbb{Z}[\beta, \beta^{-1}][\sigma^4, \sigma^{-4}][a]/(2a, a^3)$$

$k + l\alpha$

dim. $1+\alpha$ Bott element

dim $4(1-\alpha)$

dim $-\alpha$

$S^0 \to S^\alpha$

Recall

$$MU_\star = \mathbb{Z}[x_1, x_2, \ldots]$$

$$|x_i| = 2i$$

Spectral sequence for $MR_\star$:

BOTH $K\mathbb{R}$ and $M\mathbb{R}$ are complete

$$E^1 = \mathbb{Z}[x_1, x_2, \cdots][\sigma, \sigma^{-1}][a]$$

$$\dim |x_i| = i(1+\alpha) \qquad \sigma: 1-\alpha \qquad a: -\alpha$$

$$d^{2^{n+1}-1}(\sigma^{2^n}) = v_n a^{2^{n+1}-1}$$

$$v_0 = 2 \qquad v_i = x_{2^i - 1}$$

$$E^\infty = M\mathbb{R}_\star \qquad (\text{Hu-K.})$$