Note Title 1/17/2012

Equivariant & non-equivariant homotopy Theory III

Kervaire Invariant of a manifold:

Arf invariant: over $\mathbb{Z}/2$

B = bilinear form, not nec. symmetric on V

$q(x) = B(x, x)$

associated symm. bilinear form

$$\langle x, y\rangle = q(x+y) - q(x) - q(y)$$

q is a <u>quadratic refinement</u> of $\langle\, ,\, \rangle$.

Arf invariant: (characterization by Browder: element of V "vote" for $\mathrm{Arf}(q)$)

$q : V \longrightarrow \mathbb{Z}/2$

if $|q^{-1}(1)| > |q^{-1}(0)|$

$\mathrm{Arf}(q) = 1$

else, $\mathrm{Arf}(q) = 0$

<u>Framed manifolds</u>. M differentiable, compact, w/o boundary,

a <u>framing</u> on M is a stable trivialization

$$\tau_M \oplus n \xrightarrow{\cong} N$$

<u>Kervaire Invariant</u>: for $\dim(M) = 4n+2$

Poincare duality on $H^*(M, \mathbb{Z}/2)$ & $H_*(M, \mathbb{Z}/2)$.

gives pairing on $H_{2n+1}(M, \mathbb{Z}/2)$:

$$H_{2n+1}(M, \mathbb{Z}/2) \times H_{2n+1}(M, \mathbb{Z}/2) \to \mathbb{Z}/2$$

framing gives a quadratic refinement

$$q : H_{2n+1}(M, \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

$$\text{Kervaire}(M) = \text{Arf}(q)$$

question: in what dimensions are there framed manifolds of Kervaire invariant 1?

Kervaire: not in dim. 10

Browder: only possible in dim. $2^{n+1}-2$

Stable homotopy groups of spheres and framed cobordism:

framed cobordism groups $\cong \pi_* S^0$

Pontrjagin-Thom construction:

M framed

$$\nu_M \oplus \gamma \xrightarrow{\cong} N$$

$$\begin{tikzcd} S^N \arrow[r] & T(\nu_M) \arrow[r] & \Sigma^r M_+ \arrow[r] & \Sigma^r S^0 \arrow[d, equal] \\ & & & S^r \end{tikzcd}$$

collapses
complement of
tubular neighborhood

Adams spectral sequence:

based on the Steenrod algebra

i.e. algebra of cohomology operations

natural transformation

$$H^n(-, \mathbb{Z}/2) \longrightarrow H^{n+i}(-, \mathbb{Z}/2)$$

Steenrod: over $\mathbb{Z}/2$: generated by Steenrod squares

$$Sq^i : H^n \longrightarrow H^{n+i}$$

1. $Sq^0 = Id$

2. $x \in H^i$, $Sq^i(x) = x \cup x$

3 $i > \dim(x)$, $Sq^i(x) = 0$

4. <u>Cartan formula:</u>

$$Sq^n(x \cup y) = \sum_{i=0}^{n} Sq^i(x) \cup Sq^{n-i}(y)$$

5. <u>Adem relation:</u>

$$Sq^i Sq^j = \sum_{k=0}^{[i/2]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$

$\Rightarrow$ Steenrod algebra $A^* = H\mathbb{Z}/2^{\,*}\, H\mathbb{Z}/2$

dualize: $H\mathbb{Z}/2_* H\mathbb{Z}/2 = A_*$

Hopf algebras: both product & coproduct

$A_* = \mathbb{Z}/2[\xi_i]$, $|\xi_i| = 2^i - 1$

coproduct ψ:

$$\psi(\xi_i) = \sum_{j=0}^{i} \xi_j^{2^{i-j}} \otimes \xi_{i-j}.$$

Adams spectral sequence:

$$E_2 = \mathrm{Ext}^{s,t}_{A_*}(\mathbb{Z}/2, H_* X) \Rightarrow \pi_{t-s} X_2$$

X bounded below, finite type spectrum

Ext of comodules over the coalgebra A_*

<u>more naively:</u> over A^* as algebra

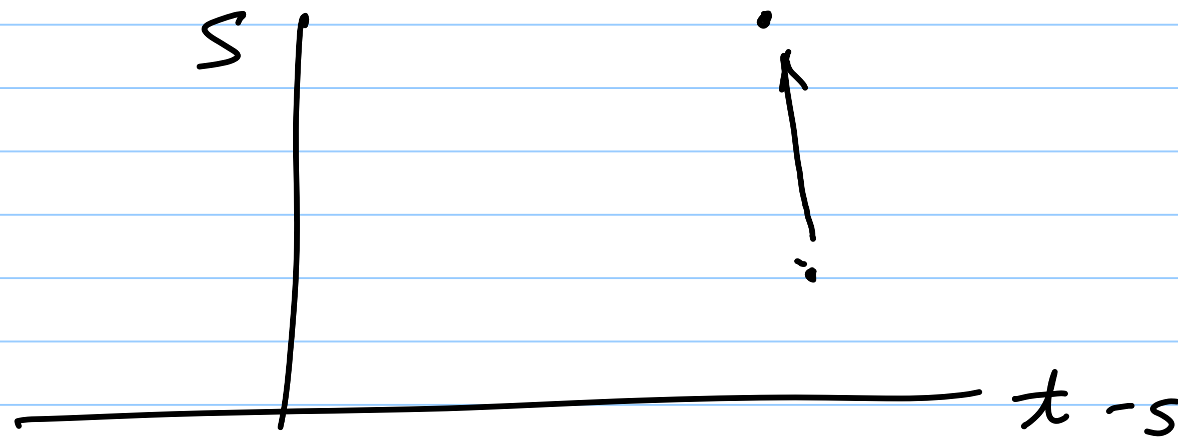
$$\mathrm{Ext}_{A^*}^{s,t}(H^*X, \mathbb{Z}/2) \Rightarrow \pi_{t-s} X_2$$

(Ext of modules)

<u>differentials:</u>

$$d_r : E_r^{s,t} \longrightarrow E^{s+r,\, t+r-1}$$

usually draw it $(t-s,\, s)$



Kervaire invariant 1 manifolds correspond to elts. in $\pi_* S^0$ which are represented in ASS E_2-term by

$$h_i^2 \in \mathrm{Ext}^{2, 2^{n+1}}$$

$h_i \in \mathrm{Ext}^{1, 2^n}$ come from primitive in A_*

x is primitive if

$$\psi(x) = x \otimes 1 + 1 \otimes x$$

in A_*, only ones are $\xi_1^{2^i} = h_i$

only h_0, h_1, h_2, h_3 survive

<u>Hopf invariant 1</u>: $\exists$ maximal number of linearly indep vector fields on S^k

or: Hopf fibration $S^{2^{n+1}-1} \longrightarrow S^{2^n}$

$(h_0 = 2)$

Kervaire invariant 1: what happens to the h_i^2's?

various people: exist up to dim 62

Hill-Hopkins-Ravenel (2009):

do not exist from dim 254 onwards