Equivariant and Non-equivariant homotopy Theory IV

Adams S. S. for spheres

\[ E_2 = \operatorname{Ext}^{s,t}_{A_*} (\mathbb{Z}/2, \mathbb{Z}/2) \Rightarrow \pi_* S^0 \]

Kervaire Inv. 1 manifold

\[ \leftrightarrow \text{elts. represented in } E_2 \]

by \( h_i^{2^i, 2^{i+1}} \)

\[ h_i \text{ do not survive to } E_\infty \text{ for } i \geq 7 \]

i.e. no Kervaire Inv. 1 manifold of dim \( 2^{n+1} - 2 \) for \( n \geq 7 \)

General plan:

construct a spectrum \( \Omega \)

from \( \mathbb{Z}/8 \)-equivariant
1. take Real cobordism $MTR$, 4 copies

$MTR \wedge MTR \wedge MTR \wedge MTR$

$MTR^{(4)}$, $\mathbb{Z}/8$-equiv.

2. invert a homotopy class $D$

of $MTR^{(4)} \Rightarrow D^{-1} MTR^{(4)}$

3. take fixed points $\Omega$
3 facts about $\Omega$:

1. detection theorem: if $\theta \in \mathbb{R}_{\leq \frac{\pi}{2}}$, then $\text{is represented by } h^2$, then they map to non-0 elts. in $\pi_{2n-2} \Omega$

2. gap theorem: $\pi_{-2} \Omega = 0$

3. periodicity theorem:

$$\pi_\ast \Omega \cong \pi_\ast + 256 \Omega$$
these three give Kervaire Inv. 1

Th.

Main Tool: slice Spectral sequence, from slice tower analogue of Postnikov tower:

given $X$, "kill" all homotopy classes $S^k \to X$ for $k > n$
at $n$-th stage
\[\xrightarrow{f_n} P^{n+1}X \xrightarrow{} P^nX \xrightarrow{} P^{n-1}X \xrightarrow{\text{attach cones}} \cdots\]

on all \(S^k \to X,\)

\(k > n\)
slice tower: equivariant analogue
replace $S^k$ by “slice cells”

$G \rtimes \Lambda^k \leq S^m \rho_k$

$K \unlhd G$ subgroup, $m \in \mathbb{Z}$,

$\rho_K$ = regular rep. of $K$

also allow $\Sigma^{-1}$ of these
for X equivariant:

\[ \ldots \rightarrow \mathbb{P}^n X \rightarrow \mathbb{P}^{n-1} X \rightarrow \ldots \]

\[ \mathbb{P}^n X = \text{homotopy fiber} \]

\[ = \text{n-th slice of } X \]

long exact sequences in

homotopy groups

\[ \Rightarrow \text{slice } \text{S.S., } RO(G)-\text{graded} \]
easier to write each "twist" separately; given virtual rep. $V$

$$E_2 = \pi_{V+\mathfrak{L}} \mathcal{P} \pi_{V+\mathfrak{L}} X \Rightarrow \pi_{V+\mathfrak{L} - s} X$$

H-H-R: for MTR, MTR$_{(2)}$, MTR$_{(4)}$

Computed what the slices are, homotopy gps.
for $X = \text{MTR}(2^n-1)$, $\mathbb{Z}/2^n$-equivariant

$n = \text{odd}: \quad P^n_n X = \ast$

$n = \text{even}: \quad P^n_n X = (\bigvee \text{slice cells}) \wedge H_{\overline{\mathbb{Z}}}^{\dim_n K \neq \{e\}}$

\underline{Gap this in this case,}

$\pi^G_\ast (\text{slice cells} \wedge H_{\overline{\mathbb{Z}}}) = 0$ i.n

$\dim -4 < i < 0$. 
**Periodicity Theorem:** look at differentials (we did for MTR)

Igor’s talk: \( d_{2k-1} (\sigma^{-2^k}) = v_k \alpha^{2^{k+1} - 1} \)

\( \alpha = \text{sign rep.} \)

\(|\alpha| = -\alpha, \quad |\sigma| = \alpha - 1, \quad |v_k| = (2^k - 1)(1 + \alpha)\)
similar els. in slice $S,S$

$E_2$-term for MTR $(2^n-1)$

1. $a \in E_2^{1,1-\alpha}$, $\alpha =$ sign rep. of $\mathbb{Z}/(2^n)$

2. $\sigma^{-2} \in E_2^{0,2-2\alpha}$

(any oriented rep. $V$, get

$w_V \in E_2^{0,1V1-V}$

$E_2$
3. $f_i \in E_2^{(g-1)i, g_i}$

$$f_{2^{k-1}} = V_k a_{2^{k-1}}$$

$$V = -2^k \alpha$$

$s$
$$d_{1+(2^k-1)}g(\sigma^{-2^k}) = a^2 \int_{2^k}^{k+1} v_k$$

Idea: invert certain multiples of $v_k$, will make powers of $\sigma^{-2^k}$ into permanent cycles once we can divide by these $v_k$'s, target of differentials from $\sigma^{-2^k} \sigma$ will be killed by
by earlier differentials
need a product of several
factors, from \( \text{MTR, MTR}^{(2)}, \text{MTR}^{(4)} \)

\[ D \in \pi_{10/8} \text{MTR}^{(4)} \]

invert \( D \Rightarrow u_{32/8} \) is perm. cycle
multiply by a factor of \( D \)

to put it into integral dim

becomes a unit in

\[ \pi_{256} G \text{D}^{-1} \text{MTR}^{(4)} \]
Detection Th. use Adams-Novikov S.S., another tool for $\pi_* S^0$:
use $MU$ instead of $H\mathbb{Z}/2$
$\text{Ext}_{MU_*MU_*} (MU_*, MU_* ) \Rightarrow \pi_* S^0$
ANSS for $S^0 \to \text{ANSS for } \to \text{homotopy fixed pt.}

$\varpi$

SS. for

$D^{-}\text{MTR}^{(4)}$

\[\downarrow\]

ASS for $S^0$

\[h^2_x\]

point: if $x$ in ANSS in $S^0 \oplus$

goes to $h^2_x$, it maps to

non-$0$ in homotopy fixed

pt. S.S. for $D^{-}\text{MTR}^{(4)}$
define $R_x = \mathbb{Z}_2 [S_8] [\sigma^\pm 1]$

$H^*(\mathbb{Z}/8, \text{D}^{-\text{MR}}) \to H^*(\mathbb{Z}/8, R_x)$

$\oplus$ computable