Equivariant and Non-equivariant homotopy Theory IV

Note Title 1/19/2012

Adams S.S for spheres

$E_2 = \mathrm{Ext}^{s,t}_{A_*}(\mathbb{Z}/2, \mathbb{Z}/2) \Rightarrow \pi_* S^0$

Kervaire Inv. 1 manifold

$\longleftrightarrow$ elts. represented in E_2

by $h_i^2 \in \mathrm{Ext}^{2, 2^{i+1}}$

Hill-Hopkins-Ravenel (2009):

h_i^2 do not survive to E_∞ for $i \geq 7$

i.e. no Kervaire Inv. 1 manifold of dim $2^{n+1}-2$ for $n \geq 7$

General plan:

construct a spectrum Ω from $\mathbb{Z}/8$-equivariant

1. take Real cobordism $M\mathbb{R}$,
   4 copies

$$\begin{tikzcd}[column sep=small] M\mathbb{R} \arrow[rr, bend right] & \wedge & M\mathbb{R} \arrow[rr, bend right] & \wedge & M\mathbb{R} \arrow[rr, bend right] & \wedge & M\mathbb{R} \arrow[llllll, bend right, "-"'] \end{tikzcd}$$

$M\mathbb{R}^{(4)}$, $\mathbb{Z}/8$-equiv.

2. invert a homotopy class D
   of $M\mathbb{R}^{(4)} \Rightarrow D^{-1} M\mathbb{R}^{(4)}$

3. take fixed points: Ω

3 facts about Ω:

1. **detection theorem**: if $\theta_i \in \pi_{2^{i+1}-2} S^0$ is represented by h_i^2, then they map to non-0 elts. in $\pi_{2^{i+1}-2}\Omega$

2. **gap theorem**: $\pi_{-2}\Omega = 0$

3. **periodicity theorem**:

$$\pi_* \Omega \cong \pi_{*+256}\Omega$$

these three give Kervaire Inv. 1 Th.

<u>Main Tool</u>: slice Spectral sequence, from slice tower

analogue of <u>Postnikov tower</u>:

given X, "kill" all homotopy classes $S^k \to X$ for $k > n$ at n-th stage

$$\begin{tikzcd}
 & & X \arrow[dll] \arrow[d, "f_n"] \arrow[drr] & & & \\
\cdots \arrow[r] & P^{n+1}X \arrow[r] & \underbrace{P^n X}_{\substack{\text{attach cones} \\ \text{on all } S^k \to X, \\ k>n}} \arrow[rr] & & P^{n-1}X \arrow[r] & \cdots
\end{tikzcd}$$

slice tower: equivariant analogue

replace S^k by "slice cells"

$$G_+ \wedge_K S^{m\rho_K}$$

$K \leq G$ subgroup, $m \in \mathbb{Z}$,

ρ_K = regular rep. of K

also allow Σ^{-1} of these

for X equivariant:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P^n X & \longrightarrow & P^{n-1} X & \longrightarrow & \cdots \\ & & \uparrow & & & & \\ & & P^n_n X & & & & \end{array}$$

$P^n_n X$ = homotopy fiber

= n-th slice of X

long exact sequences in homotopy groups

$\Rightarrow$ slice S.S., $RO(G)$-graded

easier to write each "twist"
separately: given virtual rep. V

$$E_2^{s,t} = \pi_{V+t-s} P_{|V|+t}^{|V|+t} X \Rightarrow \pi_{V+t-s} X$$

H-H-R: for $M\mathbb{R}$, $M\mathbb{R}^{(2)}$, $M\mathbb{R}^{(4)}$
computed what the slices are,
homotopy gps.

for $X = M\mathbb{R}^{(2^{n-1})}$, $\mathbb{Z}/2^n$-equivariant

$n = $ odd: $P^n_n X \simeq *$

$n = $ even: $P^n_n X = \left(\bigvee \begin{array}{c} \text{slice cells} \\ \text{of dim } n \\ K \neq \{e\} \end{array} \right) \wedge H\underline{\mathbb{Z}}$

<u>Gap th:</u> in this case,

$\pi^G_*(\text{slice cell} \wedge H\underline{\mathbb{Z}}) = 0$ in

dim $-4 < i < 0$.

Periodicity Th: look at differentials (we did for $M\mathbb{R}$)

Igor's talk: $d_{2^{k+1}-1}(\sigma^{-2^k}) = v_k\, a^{2^{k+1}-1}$

α = sign rep.

$|a| = -\alpha$, $|\sigma| = \alpha - 1$, $|v_k| = (2^k - 1)(1+\alpha)$

similar elts. in slice S.S.
E_2-term for $M\mathbb{R}^{(2^{n-1})}$

1. $a \in E_2^{1,1-\alpha}$, α = sign rep. of $\mathbb{Z}/(2^n)$

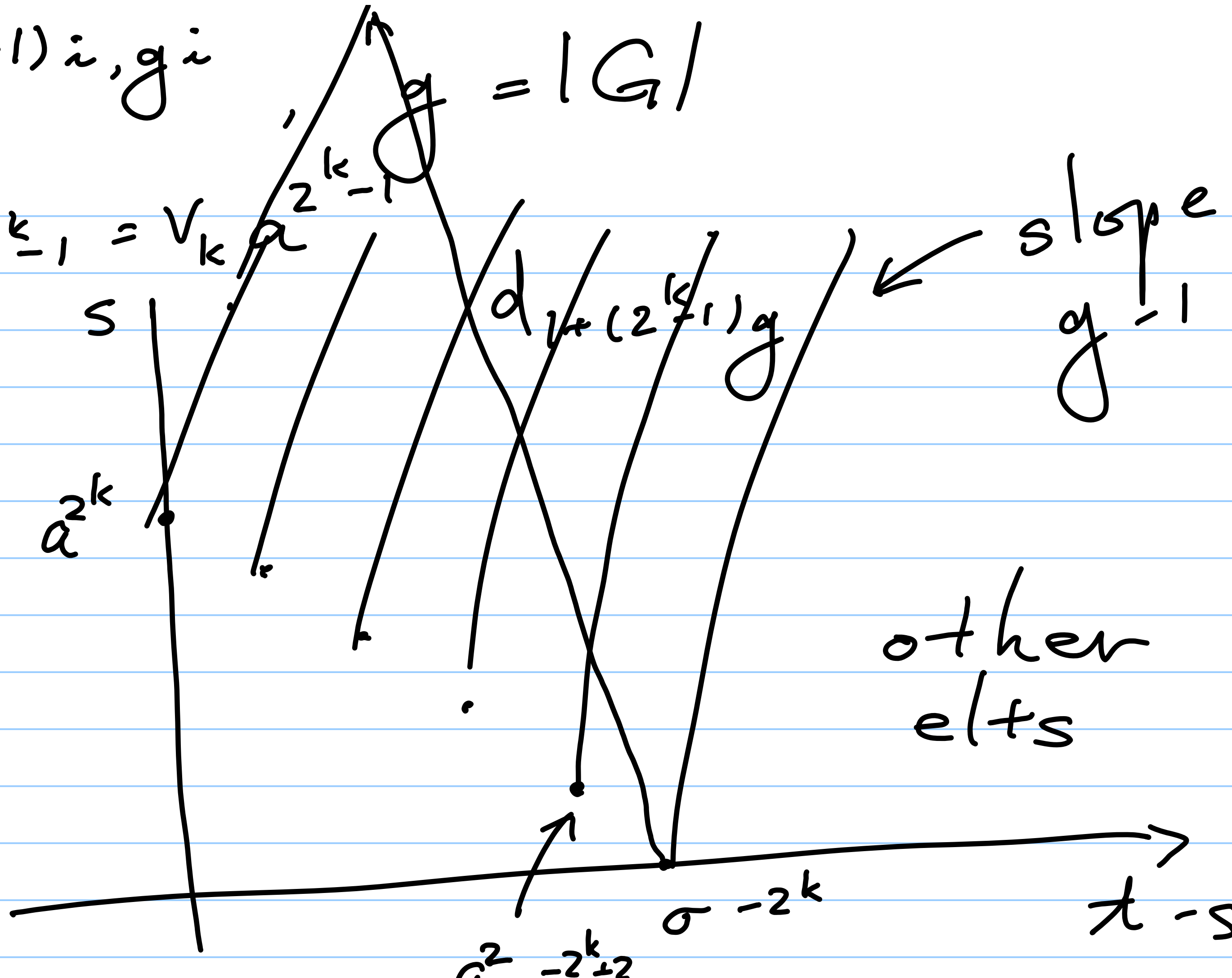
2. $\sigma^{-2} \in E_2^{0,2-2\alpha}$.

(any oriented rep. V, get
$u_V \in E_2^{0,|V|-V}$

3. $f_i \in E_2^{(g-1)i,\, gi}$, $g = |G|$

$f_{2^k-1} = v_k a^{2^k-1}$

$V = -2^k \alpha$



$$d_{1+(2^k-1)g}(\sigma^{-2^k}) = a^{2^k} f_{2^k-1}$$
$$= a^{2^{k+1}-1} v_k$$

Idea: invert certain multiples of v_k, will make powers of σ^{-2^k} into permanent cycles once we can divide by these v_k's, target of differentials from σ^{-2^k} will be killed by by earlier differentials

need a product of several factors, from MR, $MR^{(2)}$, $MR^{(4)}$

$D \in \pi_{19\rho_8} MR^{(4)}$

invert $D \Rightarrow u_{32\rho_8}$ is perm. cycle

multiply by a factor of D to put it into integral dim

becomes a unit in

$\pi^{G}_{256} D^{-1} MR^{(4)}$

Detection Th. use Adams-Novikov S.S., another tool for $\pi_* S^0$:

use MU instead of $H\mathbb{Z}/2$

$$\mathrm{Ext}_{MU_*MU}(MU_*, MU_*) \Rightarrow \pi_* S^0$$

$$\begin{tikzcd}
\text{ANSS for } S^0 \arrow[r] \arrow[d] & \text{ANSS for } \Omega \arrow[r] & \text{homotopy fixed pt. SS. for } D^{-1}M\mathbb{R}^{(4)} \arrow[d] \\
\text{ASS for } S^0 & & \oplus \\
h_i^2 & &
\end{tikzcd}$$

point: if x in ANSS in S^0 goes to h_i^2, it maps to non-0 in homotopy fixed pt. S.S. for $D^{-1}M\mathbb{R}^{(4)}$

define

8-th root of 1

$$R_* = \mathbb{Z}_2[\zeta_8][w^{\pm 1}]$$

$$H^*(\mathbb{Z}/8, D^{-1}MR^{(4)}_*) \longrightarrow \underbrace{H^*(\mathbb{Z}/8, R_*)}_{\text{computable}}$$