KAN'S COMBINATORIAL SPECTRA AND THEIR SHEAVES REVISITED

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1. Introduction

The stable homotopy category (also known as the stable category, or derived category of spectra) is a foundational setting for generalized homology and cohomology, and as such, is perhaps the most important concept of modern algebraic topology. Yet, the category does not have a canonical construction, unlike, say, the category of chain complexes, which plays an analogous role for ordinary (co)homology. In contrast, different approaches to the stable category exist, each of which has some advantages and some disadvantages. An extensive foundational and calculational treatment of the stable category was given by Adams [1]. But Adams' treatment does not give an underlying "point set category", which is often needed in constructions, just as actual chain complexes, and not just the objects of their derived category, are needed in homological algebra. A point set level category of spectra, very closely analogous to the category of topological spaces, is provided by *May spectra* [17], which, notably, also works equivariantly for compact Lie groups. The May category has a number of more recent improvements, many of which are related to constructing a point set level commutative associative smash product [8, 13, 19].

A completely different construction of the stable homotopy category can be given using a concept of a *combinatorial spectrum* discovered much earlier by Kan [16], which can be described as a "naive stabilization" of a simplicial set. While very appealing aesthetically, this approach has not had nearly as much follow-up as constructions based on topological spaces. A part of the reason is that even defining a smash product of a Kan spectrum with a based simplicial set (which is necessary in treating generalized homology of spaces) is difficult, due

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to the fact that the smash product of based simplicial sets does not commute with suspension of combinatorial spectra.

Yet, combinatorial spectra have some advantages. Notably, K.S. Brown [4] developed a fully functional theory of sheaves of combinatorial spectra which is a generalization of abelian sheaves, and can be used to define generalized sheaf cohomology. Brown's category of sheaves of combinatorial spectra was also used by Piacenza [21] to treat *locally constant sheaves of spectra*, which is an approach to *parametrized spectra*. A rigorous definition of the derived category of parametrized spectra was a notoriously hard problem. Treatments based on May spectra were more recently given in [14] and [20], and those also work for compact Lie groups. The construction [21] can be used to construct the derived category of parametrized spectra as a full subcategory of the derived category of sheaves of combinatorial spectra. On the other hand, a fully functional category of sheaves of May spectra, beyond locally constant, has so far not been constructed. Perhaps the difficulty with the smash product of combinatorial spectra is heuristically related to the ease with which they are sheafified: for example, left derived functors are less natural in abelian sheaves also, since abelian sheaves do not have enough projectives.

In [5], Brown and Gersten applied the results of [4] to algebraic K-theory, which was later used by Thomason [25]. Most of the discussion of sheaf theory concepts for spectra since that time used Thomason's approach (see e.g. the survey paper [9] for examples). Thomason noticed that given certain hypotheses on the site, one can mostly get by with presheaves, by using cosimplicial Godement resolutions, which can be constructed on the level of presheaves (since they only use stalks). For Godement resolutions, one only needs a category with directed colimits and products. Applying these techniques, Thomason [25] used the category of presheaves of "fibrant simplicial (pre)spectra" of Bousfield and Friedlander [2] to define his version of generalized sheaf cohomology. The category [2] is not canonical, many variants give the same result. In general, however, any presheaf approach to sheaf theory is somewhat "Ersatz": for example, it does not have full functoriality with respect to functors which cannot be computed on presheaves, such as the direct image. Such examples, using the original Brown theory [4, 5], occurred in the work of Gillet on the Riemann-Roch theorem in K-theory [10, 11].

In the present paper, we revisit Kan's combinatorial spectra, and their sheaves, in view of the new axiomatic approach to homotopy theory given in [12], (see also [23]). The prevailing approach to the foundations of homotopy theory at this point is *Quillen model structure* [22]. A Quillen model structures gives a computable construction of the derived category, and access to left and right derived functors for certain functors known as left and right Quillen adjoints. Describing a Quillen model structure on a category has become the standard method of constructing a derived category in homotopy theory. Yet, constructing Quillen structures can be often technical and non-canonical: Different model structures may exist, which may, for example, describe the same derived category, but may differ in Quillen adjoints, so different model structures may actually be needed to making different functors derived.

The paper [12] formalized in general axiomatic terms an approach to constructing derived categories which preceeded Quillen model structures. This approach was first used in 1949 by J.H.C.Whitead [26, 27] to develop the derived category of spaces, and was also used to construct the stable homotopy category from May spectra [17], although in both cases, model structures exist also. In 1956, Cartan and Eilenberg [6] used Whitehead's approach, and its dual, to construct *Ext* groups in abelian categories with enough projectives, resp. enough injectives. In [12], this construction was formalized in a general context. For a category $\mathscr{C}$ and a class of morphisms in $\mathscr{D} \subseteq \mathscr{C}$, by the *derived category* (if one exists) we shall mean the universal category (on the same class of objects) in which the morphisms in $\mathscr{D}$ become isomorphisms.

Definition [12]: Suppose $\mathscr{C}$ is a category together with two classes of morphisms $\mathscr{S} \subseteq \mathscr{E}$ called *strong and weak homotopy equivalences.* For simplicity, we shall assume that both $\mathscr{S}$ and $\mathscr{E}$ contain all isomorphisms, and satisfy the 2/3 axiom (i.e. in a commuting triangle of morphisms, if two out of three morphisms are in the class, so is the third). Suppose further that the derived category of $\mathscr{C}$ with respect to strong homotopies exists. (We call it the *strong homotopy category.*)

Suppose now there is a class $\mathscr{B} \subseteq Obj(\mathscr{C})$ such that for any $y \in \mathscr{B}$, any weak equivalence $f : x_1 \to x_2$ induces a bijection on sets of morphisms into y (resp. from y) in the strong homotopy category. (In that case, we say that $\mathscr{B}$ is *local* (resp, *co-local*) with respect to weak equivalences in the strong homotopy category.) Suppose further that for every $x \in Obj(\mathscr{C})$, there exists a weak equivalence $x \to x'$ (resp. $x' \to x$). Then we call the category $\mathscr{C}$ together with the data just specified a *right (resp. left) Cartan-Eilenberg category.*

The authors of [12] prove that for any right or left Cartan-Eilenberg category, a derived category with respect to weak homotopy equivalences (called the *weak homotopy category*) exists, and is equivalent to the full subcategory of the strong homotopy category on $\mathscr{B}$. In addition to the fact that a structure of a left or right Cartan-Eilenberg category is often technically easier to prove than a Quillen model structure, it also has some advantages. For example, it gives a construction of a left (resp. right) derived functor of *any* functor which preserves strong homotopy equivalence. Therefore, the more restrictive the choice of strong homotopy equivalences is, the stronger the Cartan-Eilenberg structure becomes. (Note that it is possible to choose strong and weak equivalences to be the same, in which the structure becomes trivial.) The most geoetric situation is when strong homotopy equivalence is actual homotopy equivalence with respect to some "naive" concept of homotopy (such as, for example, homotopy of spaces, simplicial sets or spectra, or chain homotopy).

While no direct relation of Cartan-Eilenberg structure with Quillen model structure is known in either direction, heuristicaly, when both structures are present, one expects a Quillen model category to be left (resp. right) Cartan-Eilenberg when every object is fibrant (resp. cofibrant). A very important example is the case of spaces where the category of topological spaces is (by Whitehead [26, 27]) left Cartan-Eilenberg, while the category of simplicial sets is right Cartan-Eilenberg ([12]). This can be considered to be a manifestation of Eckmann-Hilton duality. Generally, right Cartan-Eileneberg categories are suitable for sheafification, just as one can develop a good derived category of sheaves on an abelian category with enough injectives.

In the case of spectra, which are a middle ground between spaces and abelian categories, May spectra (and also the S-modules of [8]) are left Cartan-Eilenberg with respect to homotopy equivalences (while the symmetric spectra of [13] are neither left nor right Cartan-Eilenberg). The left Cartan-Eilenberg property turns out to be very valuable in imitating algebraic structures on spectra [15], but it is not suitable for a fully functorial sheaf theory. A right Cartan-Eilenberg structure is needed. In [23], such a structure was found on presheaves of Thomason spectra where the strong homotopy equivalences there are section-wise weak equivalences of presheaves.

This begs the following question: *Is there a right Cartan-Eilenberg theory of spectra (Eckmann-Hilton dual to May spectra), and a right*

Cartan-Eilenberg theory of their sheaves, where strong equivalences are naive homotopy equivalences (defined by actual homotopies)?

The main purpose of this paper is to answer these questions in the affirmative. However, it appears hopeless to use any variation of the construction [2] for this purpose. This is, roughly speaking, because in a stable Quillen model structure, an arbitrary prespectrum cannot be cofibrant – one needs its structure maps to be cofibrations. On the other hand, Kan's combinatorial spectra do work. We prove that Kan's combinatorial spectra are right Cartan-Eilenberg with respect to homotopy equivalence (Corollary 14). We also prove that over a sufficiently nice site, the category of sheaves of combinatorial spectra is right Cartan-Eilengerg with respect to homotopy equivalences (Theorem 26). In fact, this does not seem to be in the literature even for sheaves of simplicial sets, so we also prove that first (Theorem 21).

The present paper is organized as follows: In Section 2, we review Kan's construction of combinatorial spectra [16], and develop some additional technical concepts. We also prove that they are right Cartan-Eilenberg with respect to homotopy equivalence. In Section 3, we make some observations on cosimplicial realization which we need later. In Section 4, we discuss "nice" sites and prove right Cartan-Eilenberg property with respect to homotopy equivalence for sheaves of simplicial sets and combinatorial spectra.

2. Combinatorial spectra

Combinatorial spectra were introduced by D. Kan in [16]. Recall the simplicial category Δ whose object set is $\mathbb{N}_0$ and $\Delta(m,n)$ is the set of maps

$$\rho : \mathbf{m} = \{0,\dots,m\} \to \mathbf{n} = \{0,\dots,n\} \tag{1}$$

preserving $\leq$. There is a self-functor

$$\Phi : \Delta \to \Delta$$

where $\Phi(m) = m+1$ and for ρ as in (1), $\Phi(\rho)$ coincides with ρ on $\mathbf{m}$, and

$$(\Phi(\rho))(m+1) = n+1.$$

The category Δ_{st} is the (strict) colimit in the category of small categories of the diagram

$$\Delta \xrightarrow{\Phi} \Delta \xrightarrow{\Phi} \dots \tag{2}$$

Therefore, one can identify the category Δ_{st} with the category whose object set is $\mathbb{Z}$, and morphisms are generated by "faces" $d_i : m \to m+1$

and "degeneracies" $s_i : m \to m-1$ which satisfy the usual simplicial relations.

Denote by $\Phi^{\infty-n}$ the inclusion of the n'th term Δ of (2) into the colimit Δ_{st}. Note that one can have $n \in \mathbb{Z}$. Also note that we can identify $\Delta_{st}(k,\ell)$ with the set of $\leq$-preserving maps $f : \mathbb{N}_0 \to \mathbb{N}_0$ which are of the form

$$f = \Phi^{\infty-n}(\alpha) \tag{3}$$

for some $\alpha \in \Delta(k+n, \ell+n)$ where (3) is defined as the extension of α given by

$$\Phi^{\infty-n}(\alpha)(s) = \ell - k + s \text{ for } s > k+n$$

(i.e., put in another way, which satisfy $f(s+1) = f(s)+1$ for s large enough). From this point of view, *faces* (resp. *degeneracies*) in the wider sense are morphisms in Δ_{st} which are injective (resp. onto) as maps $\mathbb{N}_0 \to \mathbb{N}_0$. These are, of course, precisely those morphisms which are compositions of d_i's (resp. of s_i's), $i \geq 0$.

One denotes by $Set_\bullet$ the category of based sets, whose objects are based sets (sets with a distinguished base point $*$), and morphisms are mappings preserving $*$. Consider the category Δ^{Op}-$Set_\bullet$ of based simplicial sets, which is the category of functors $\Delta^{Op} \to Set_\bullet$ and natural transformations. Then there is a functor

$$\Omega^k : \Delta^{Op}\text{-}Set_\bullet \to \Delta^{Op}\text{-}Set_\bullet$$

where for a simplicial set $T : \Delta^{Op} \to Set_\bullet$,

$$(\Omega^k(T))(n) = \{x \in T(n+k) \mid d_{n+1}(x) = \cdots = d_{n+k}(x) = *\}$$

and for $x \in (\Omega^k(T))(n)$, the operators s_i, d_i, $j \leq n$ act on x the same way as in T. One writes Ω instead of Ω^1, and one has

$$\Omega^{k+\ell} = \Omega^k \Omega^\ell.$$

The functor Ω^k has a left adjoint denoted by Σ^k.

We can describe the functor Σ explicitly as follows (a similar description also holds for Σ^k): Let Δ_0 be the subcategory of Δ consisting of the same objects and the morphisms (1) such that

$$|\rho^{-1}(n)| \leq 1. \tag{4}$$

(Note that the category Δ_0 contains the image of Φ, and morphisms in the image of Φ are precisely those which satisfy the inequality in (4).) Then we have a functor

$$\Sigma_0 : \Delta^{Op}\text{-}Set_\bullet \to \Delta_0^{Op}\text{-}Set_\bullet$$

where $\Sigma_0 Z(n) = Z(n-1)$ for $n \geq 1$ and $\Sigma_0(Z)(0) = *$, and on $\Sigma_0 Z$, morphisms of the form $\phi(\rho)$ act the same way as ρ on Z, and other morphisms act by $*$. If we denote the inclusion functor $\iota : \Delta_0^{Op} \to \Delta^{Op}$ and its left Kan extension by $\iota_\sharp$, then

$$\Sigma = \iota_\sharp \Sigma_0.$$

D.Kan [16] proves the following

Proposition 1. *The functor Σ^k commutes with the simplicial realization functor $|?|$ up to canonical natural isomorphism, where Σ^k on based spaces denotes the canonical suspension $? \wedge S^k$.*

Proof. It suffices to consider $k = 1$. Consider the case of the standard n-simplex Δ_{n+} where Δ_n is the representable simplicial set, $\Delta_n(k) = \Delta(k,n)$. Then $\Sigma(\Delta_{n+})$ has a non-degenerate element $x \in \Sigma(\Delta_{n+})(n+1)$ which satisfies $d_{n+1}(x) = d_0^n(x) = *$. Clearly, the geometric realization of this is canonically identified with $\Sigma|\Delta_{n+}|$, and the identification is compatible with faces and degeneracies, and thus applies canonically and naturally to every simplicial set. □

Corollary 2. *We have a natural inclusion*

$$|\Omega(X)| \subseteq \Omega|X| \tag{5}$$

on based simplicial sets X.

Proof. We have a map given by simplicial realization of the counit of adjunction:

$$\Sigma|\Omega(X)| = |\Sigma\Omega(X)| \to |X|.$$

The map (5) is its adjoint. One easily verifies that it is an inclusion. □

Proposition 3. *For a based simplicial set Z, the unit of adjunction*

$$\eta : Z \to \Omega\Sigma Z \tag{6}$$

is an isomorphism.

Proof. Every element of a simplicial set is uniquely expressible as an iterated degeneracy (i.e. a composition of degeneracies) of a non-degenerate element (i.e. one which is not in the image of a degeneracy). Now we have a bijection b from the non-degenerate elements of $Z(n)$ to the non-degenerate elements of $\Sigma Z(n+1)$ for $n \geq 0$ (since this is by definition true for Σ_0, and $\iota_\sharp$ is an inclusion on each $?(n)$, and all of the elements in its image are degenerate.

On the other hand, we also have a bijection between the non-degenerate elements of $\Omega T(n)$ to the non-degenerate elements of $T(n+1)$ which satisfy $d_{n+1}(x) = *$ for any based simplicial set T. This is because if, for $x \in T(n+1)$, $d_{n+1}(x) = *$, then $x = \rho(y)$ for y non-degenerate where

ρ is an iterated degeneracy satisfying (4) (with n replaced by $n+1$). Thus, if $x \in \Omega T(n)$ is non-degenerate if and only if $x \in T(n+1)$ is.

Now if we denote for any simplicial set Z by Z_{nd} the sequence of sets of non-degenerate elements, we see that for $x \in Z_{nd}(n)$, $b(x) \in (\Sigma Z)_{nd}(n+1)$ satisfies $d_{n+1}(b) = *$. Thus, η preserves non-degenerate elements, and we have a commutative diagram

$$\begin{tikzcd}
Z_{nd}(n) \arrow[rr, "\cong"] \arrow[dr, "\eta_{nd}"'] & & (\Sigma Z)_{nd}(n+1) \\
& (\Omega\Sigma Z)_{nd}(n). \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

Thus, η_{nd} is a bijection, which implies the statement of the Proposition.

□

Similarly, one can consider the category Δ_{st}^{Op}-$Set_\bullet$ of functors $\Delta_{st}^{Op} \to Set_\bullet$ and natural transformations, called the category of stable simplicial sets. One has functors

$$\Omega^{\infty+k} : \Delta_{st}^{Op}\text{-}Set_\bullet \to \Delta^{Op}\text{-}Set_\bullet$$

where for

$$(\Omega^{\infty+k}(T))(n) = \{x \in T(n+k) \mid d_m(x) = * \text{ for } m > n\}.$$

Clearly, we have

$$\Omega^\ell \Omega^{\infty+k} = \Omega^{\infty+k+\ell},$$

and the functor $\Omega^{\infty+k}$ has a left adjoint $\Sigma^{\infty+k}$.

The category $\mathscr{S}$ of *combinatorial spectra* is the full subcategory of Δ_{st}^{Op}-$Set_\bullet$ on all stable simplicial sets T with the property that for all $x \in T(n)$ there exists a k such that $d_m(x) = *$ for $m > k$. This is a coreflexive subcategory of Δ_{st}^{Op}-$Set_\bullet$ (by passing to the subset of all elements satisfying the condition). Therefore, the category $\mathscr{S}$ has all limits and colimits.

Proposition 4. *The category $\mathscr{S}$ is canonically equivalent to the category whose objects are sequences of simplicial sets $(Z_n)_{n\in\mathbb{N}_0}$ ($\mathbb{N}_0$ can also be equivalently replaced with $\mathbb{Z}$) together with isomorphisms of simplicial sets*

$$\rho_n : Z_n \xrightarrow{\cong} \Omega Z_{n+1} \tag{7}$$

and morphisms are sequences of morphisms of simplicial sets commuting with the structure maps.

Proof. For a combinatorial spectrum Z, put

$$Z_n = \Omega^{\infty-n} Z.$$

For a sequence Z_n with structure maps (7), define

$$Z(n) = \operatorname*{colim}_k Z_k(n+k).$$

By definition, these functors are inverse to each other up to canonical natural isomorphisms. □

In view of Corollary 2, Proposition 4 gives the sequence $(|Z_n|)$ a functorial structure of an inclusion prespectrum. Denote by $\mathscr{L}(Z)$ the associated May spectrum. A morphism $f : X \to Y$ of combinatorial spectra is called a *(weak) equivalence* if $\mathscr{L}(f)$ is an equivalence of May spectra.

Also, Proposition 4 suggests to also consider the concept of a *combinatorial prespectrum* Z which is a sequence of morphisms of based simplicial sets

$$\rho_n : Z_n \to \Omega Z_{n+1},\ n \in \mathbb{Z} \tag{8}$$

(or, by adjunction equivalently, $\Sigma Z_n \to Z_{n+1}$), without any additional assumptions on ρ_n. A morphism of combinatorial prespectra $f : Z \to T$ is a sequence of morphisms $f_n : Z_n \to T_n$ such that we have commutative diagrams

$$\begin{tikzcd}
Z_n \arrow[r, "f_n"] \arrow[d, "\rho_n"'] & T_n \arrow[d, "\rho_n"] \\
\Omega Z_{n+1} \arrow[r, "\Omega f_{n+1}"] & \Omega T_{n+1}.
\end{tikzcd}$$

Denote the category of combinatorial prespectra by $\mathscr{P}$. We then have a "forgetful" functor

$$Ps : \mathscr{S} \to \mathscr{P},$$

which has a left adjoint

$$Sp : \mathscr{P} \to \mathscr{S}.$$

In the notation of Proposition 4, for a prespectrum Z, we have

$$(Sp(Z))_n = \operatorname{colim} \Omega^k Z_{n+k}.$$

Similarly as in the context of May spectra, a variant of the construction of $\mathscr{P}$ is obtained by replacing the indexing set $\mathbb{Z}$ with $\mathbb{N}_0$ in (8). The resulting category will be denoted by $\mathscr{P}_0$ and the analogues of the functors Ps, Sp by Ps_0, Sp_0. There is a canonical forgetful functor $\mathscr{P} \to \mathscr{P}_0$, but it is not an equivalence of categories.

For a combinatorial spectrum X, an element $x \in X(n)$, $x \neq *$, is called *non-degenerate* if x is not in the image of a degeneracy.

Lemma 5. *Let Z be a combinatorial spectrum and let $x \in Z(n)$, $x \neq *$. Then there exists a unique degeneracy in the wider sense $s \in \Delta_{st}(n,m)$ and a unique element $y \in Z(m)$ such that $x = s(y)$ and y is non-degenerate.*

Proof. Suppose $d_s(x) = *$ for $s > N$. By the proof of the previous Lemma, we may consider x as an element of $Z_{n-N}(N)$ which is an ordinary (based) simplicial set, where this fact is well known, existence and uniqueness. N is of course not uniquely determined, but if also $x = s'(y')$ for a degeneracy s' and a non-degenerate element y', we can use the larger of both N's to see that $y = y'$, $s = s'$. □

The sphere spectrum S is the free combinatorial spectrum on one element $\alpha \in S(0)$ with the relation that $d_i(\alpha) = *$ for $i \geq 0$. We have

$$S(n) = \{\alpha_n, *\} \text{ for } n \geq 0$$

where α_n is the iterated degeneracy of α, and

$$S(n) = \{*\} \text{ for } n < 0.$$

It is immediate from the definition that By commutation of adjoints, we have

$$S = \Sigma^\infty S^0.$$

In the notation of Proposition 4, the based simplicial sets S_n are the free based simplicial sets on one element $\alpha_n \in S_n(n)$ such that $d_i(\alpha_n) = *$ for all $i \geq 0$. In particular, we have

$$\Omega S_n \cong S_{n-1}.$$

Recall that the standard n-simplex is the based simplicial set

$$\Delta_n = \Delta(?, n)$$

(the representable functor). As usual, we denote by Δ_n° the simplicial set obtained from Δ_n by deleting the non-degenerate element $\alpha \in \Delta_n(n)$ and all its degeneracies, and by $V_{n,k}$ the simplicial set obtained by deleting, additionally, $d_k(\alpha)$ and all its degeneracies.

A *relative combinatorial cell spectrum* is a morphism $f : X \to Y$ of combinatorial spectra such that there exist combinatorial spectra $Y_{(m)}$, $m \geq -1$ where $Y_{(-1)} = X$, indexing sets I_m, indexing morphisms

$n_m : I_m \to \mathbb{N}_0$, $\ell_m : I_m \to \mathbb{Z}$ and pushout diagrams of combinatorial spectra

$$\begin{tikzcd} \bigvee_{i\in I_m} \Sigma^{\infty+\ell_m(i)}\Delta^{\circ}_{n_m(i)+} \arrow[r, "f_m"] \arrow[d, "\subseteq"'] & Y_{(m-1)} \arrow[d] \\ \bigvee_{i\in I_m} \Sigma^{\infty+\ell_m(i)}\Delta_{n_m(i)+} \arrow[r] & Y_{(m)} \end{tikzcd} \tag{9}$$

such that

$$Y = \lim_{\to} Y_{(m)}.$$

Here $\bigvee$ denotes the coproduct.

An *anodyne extension* is defined in the same way as a relative combinatorial cell spectrum where in (9), $\Delta^{\circ}_{n_m(i)}$ is replaced by $V_{n_m(i),k_m(i)}$. Clearly, anodyne extensions are weak equivalences.

Proposition 6. *Every injective morphism of combinatorial spectra is a relative combinatorial cell spectrum. In particular, every combinatorial spectrum is a cell combinatorial spectrum.*

Proof. The only non-degenerate element of $\Sigma^{\infty+\ell}\Delta_{n+}$ which is not in $\Sigma^{\infty+\ell}\Delta^{\circ}_{n+}$ is in dimension $n+\ell$, corresponding to the non-degenerate element of Δ_n which is not in Δ°_n. For a non-degenerate element $x \in Y(n) \smallsetminus X(n)$, let the *degree* of x be the minimum k such that $d_i(x) = *$ for $i > k$. Then we can let $Y_{(k)}(n)$ be the set of all elements x of the form $s(y)$ where s is a degeneracy in the wider sense, and y is non-degenerate of degree $\leq k$. By Lemma 5, $Y_{(k)}$ is a subspectrum of Y, and moreover, $Y_{(k+1)}(n)$ is obtained from $Y_{(k)}$ by attaching a cell $\Sigma^{\infty-n}(\Delta_{k+})$ for every non-degenerate element of $Y(n)$ of degree k. $\square$

It is important to note that if we can attach a cell of the form $\Sigma^{\infty-\ell}\Delta_{n+}$ to a combinatorial spectrum, we can obtain an *isomorphic* combinatorial spectrum by attaching a cell of the form $\Sigma^{\infty-\ell-k}\Delta_{(n+k)+}$ instead for any $k \geq 0$. Also, a combinatorial cell spectrum (and hence every combinatorial spectrum) is naturally a directed colimit of inclusions of its finite cell subspectra, which are, by adjunction, shift desuspensions of simplicial sets. Therefore, in particular, every combinatorial spectrum is a directed direct limit of inclusions of shift desuspensions.

For based simplicial sets K, T, we have a simplicial set

$$F(K,T) = \Delta^{Op}\text{-}Set_{\bullet}(K \wedge (\Delta_n)_+, T)$$

where $?_+$ means attaching a disjoint base point. The left adjoint to this functor is $K \wedge ?$.

Lemma 7. [16] *There is a canonical natural morphism of based simplicial sets*

$$(\Sigma K) \wedge T \to \Sigma(K \wedge T) \tag{10}$$

which is a weak equivalence (i.e. becomes a homotopy equivalence after applying simplicial realization).

Proof. Consider again the case when both $K = \Delta_{m+}$, $T = \Delta_{n+}$. The non-degenerate elements in dimension $m+n$ of the simplicial set

$$\Delta_{m+} \wedge \Delta_{n+} = (\Delta_m \times \Delta_n)_+ \tag{11}$$

correspond to shuffles of m ordered elements and n other ordered elements, i.e. their number is

$$\binom{m+n}{m}.$$

After suspension, there will be the same number of nondegenerate elements in dimension $m+n+1$. In

$$(\Sigma\Delta_{m+}) \wedge \Delta_{n+}, \tag{12}$$

on the other hand, the non-degenerate elements in dimension $m+n+1$ correspond to shuffles of $m+1$ ordered elements and n ordered elements (same is if we replaced $\Sigma\Delta_{m+}$ by $(\Delta_{m+1})_+$, i.e. their number is

$$\binom{m+n+1}{n}.$$

The morphism from (12) to the suspension of (11) is obtained by applying the last degeneracy (s_{m+1}) to all non-degenerate elements of dimension $m+n+1$ corresponding to the shuffles of the $m+1$ and n ordered elements where the last of the $m+1$ elements is not in the end. One verifies that this recipe is natural in the simplicial category. Additionally, by definition, after simplicial realization, the map (10) becomes a quasi-fibration [7] with contractible fibers, so it is a weak equivalence between CW-complexes, and hence a homotopy equivalence. □

It is well known that if a functor $F_1 : C \to D$ has a right adjoint G_1 and a functor $F_2 : C \to D$ has a right adjoint G_2, and we have a natural transformation $F_1 \to F_2$, then we have a natural map of sets

$$D(F_2X, Y) \to D(F_1X, Y),$$

which is, by adjunction,

$$C(X, G_2Y) \to C(X, G_1Y)$$

which, by the Yoneda lemma, gives a canonical natural transformation

$$G_2 \to G_1.$$

Applying this principle to the situation of Lemma 7, the right adjoint to $F_1 = (\Sigma ?) \wedge T$ is $G_1 = \Omega F(T, ?)$, and the right adjoint to $F_2 = \Sigma(? \wedge T)$ is $G_2 = F(T, \Omega(?))$, so we get a canonical natural transformation

$$F(T, \Omega Y) \to \Omega F(T, Y) \tag{13}$$

which, moreover, is injective, since it is right adjoint to a surjective map.

This means that if Z is a combinatorial prespectrum and T is a based simplicial set, then we get canonical morphisms

$$F(T, Z_n) \to \Omega F(T, Z_{n+1}), \tag{14}$$

and since Ω commutes with colimits of sequences, a combinatorial prespectrum, denoted by $F_p(T, Z)$. We may of course go on to define a spectrum $F(T, Z)$ by

$$F(T, Z) = Sp(F_p(T, Z)),$$

i.e. by setting

$$F(T, Z)_n = \operatorname*{colim}_k \Omega^k F(T, Z_{n+k}). \tag{15}$$

This seems particularly natural for a combinatorial spectrum Z, where one sees that the morphisms (14) are inclusions.

It is important to note, however, that the functor $F_p(T, ?) : \mathscr{P} \to \mathscr{P}$ has a left adjoint $T \cdot ?$, while the functor $F(T, ?)$ does not. For a prespectrum Z and a based simplicial set T, the based simplicial set $(T \cdot Z)_n$ is the colimit of the diagram of based simplicial sets

$$\begin{tikzcd}
 & & \vdots \arrow[d] \\
 & \Sigma(T \wedge \Sigma Z_{n-2}) \arrow[r] \arrow[d] & \Sigma^2(T \wedge Z_{n-2}) \\
T \wedge \Sigma Z_{n-1} \arrow[r] \arrow[d] & \Sigma(T \wedge Z_{n-1}) & \\
T \wedge Z_n & &
\end{tikzcd} \tag{16}$$

An analogue of these constructions also exists when replacing $\mathscr{P}$ with $\mathscr{P}_0$. It is interesting to note that in that case, the diagram (16) is finite for each n, containing only the terms involving Z_{n-k} for $k \leq n$.

In Diagram (16), the horizontal morphisms are weak equivalences of simplicial sets, but unless we know something about the vertical arrows, unfortunately this does not appear to imply anything about the colimit of (16). On the other hand, if Z is a combinatorial spectrum, the vertical arrows of (16) are injective, so the inclusion of $T \wedge Z_n$ into the colimit is a weak equivalence. Inspecting non-degenerate elements in (16), we obtain the following

Lemma 8. *If Z is a combinatorial spectrum, then $? \star Z = Sp? \cdot Ps(Z)$ preserves weak equivalences as well as injective morphisms in the $?$ coordinate.*

□

For a combinatorial spectrum Z and a based simplicial set K, it is possible to describe $K \star Z$ explicitly in terms of the cell decomposition of Z given in the proof of Proposition 6. We define $K \star Y_{(k)}$ by induction on k so that for every non-degenerate cell

$$e : \Sigma^{\infty-n}\Delta_{\ell+} \to Y_{(k)} \tag{17}$$

of degree $\ell \leq k$, we have a morphism

$$K \star e : \Sigma^{\infty-n}K \wedge \Delta_{\ell+} \to K \star Y_{(k)} \tag{18}$$

such that for an iterated face $d : \Delta_m \to \Delta_\ell$, if we denote by e' the non-degenerate cell of $Y_{(k)}$ of degree $p \leq m$, we have a commutative diagram

$$\begin{tikzcd}
\Sigma^{\infty-n}\Sigma^{m-p}K \wedge \Delta_{p+} \arrow[r, "K\star e'"] & Y_{(k)} \\
\Sigma^{\infty-n}K \wedge \Sigma^{m-p}\Delta_{p+} \arrow[u] & \\
\Sigma^{\infty-n}K \wedge \Delta_{m+} \arrow[u] \arrow[r, "\Sigma^{\infty-n}K\wedge d"'] & \Sigma^{\infty-n}K \wedge \Delta_{\ell+} \arrow[uu, "K\star e"']
\end{tikzcd} \tag{19}$$

where the left column is projection followed by an iteration of the morphism (10) of Lemma 7. We may start with $k = -1$, $Y_{(-1)} = *$. We have $K \star Y_{(-1)} = *$. Assuming $K \star Y_{(k)}$ has been defined, and e is a cell (17) of $Y_{(k+1)}$ of degree $\ell = k + 1$, we have an attaching map

$$\Sigma^{\infty-n}K \wedge \Delta^{\circ}_{\ell+} \to K \star Y_{(k)} \tag{20}$$

given by gluing (18) with e replaced by the faces of e, using Diagram (19). Thus, we may push out (20) with the inclusion

$$\Sigma^{\infty-n}K \wedge \Delta^{\circ}_{\ell+} \to \Sigma^{\infty-n}K \wedge \Delta_{\ell+}.$$

Note that this description can also be taken as a definition of $K \star Z$. Functoriality follows by applying iterations of the morphisms (10), since morphisms of combinatorial spectra do not increase the degree of cells. Immediately from this construction, there follows

Lemma 9. *We have a natural (in all coordinates) isomorphism of spectra*

$$(T_1 \wedge T_2) \star Z \cong T_1 \star (T_2 \star Z) \tag{21}$$

□

Note that $S^0 = *_+$ (i.e. the simplicial set of two points with one of them as base point) is a left unit for the operation ?·?, hence also for $\star$. Now denoting $I = \Delta_1$, a *homotopy* $h : f \simeq g$ two morphisms of combinatorial spectra

$$f, g : X \to Y \tag{22}$$

is a morphism

$$h : I_+ \star X \to Y \tag{23}$$

such that $hd_0 = f$, $hd_1 = g$. The equivalence relation of homotopy $\simeq$ on $\mathscr{S}(X,Y)$ is detined as the smallest equivalence relation containing the relation of existence of a homotopy $h : f \simeq g$. By functoriality of $\star$, this is a congruence relation on the category of combinatorial spectra, and the corresponding quotient category is denoted by $h\mathscr{S}$, and called the *(strong) homotopy category* of combinatorial spectra. An isomorphism in $h\mathscr{S}$ is called a *homotopy equivalence.*

Lemma 10. *The inclusions*

$$d_i : X = S^0 \star X \to I_+ \star X$$

are injective morphisms, and weak equivalences. A homotopy equivalence of combinatorial spectra is a weak equivalence.

Proof. The second statement clearly follows from the first. The first statement follows from the fact that $d_i : S^0 \to I_+$ are injective morphisms and weak equivalences. □

A morphism $f : X \to Y$ of combinatorial spectra is called a *Kan fibration* if in the following commutative diagram, for any choice of horizontal arrows, the diagonal arrow exists:

$$\begin{tikzcd} \Sigma^{\infty+\ell}(V_{n,k})_+ \arrow[r] \arrow[d, "\subset"'] & X \arrow[d, "f"] \\ \Sigma^{\infty+\ell}\Delta_{n+} \arrow[r] \arrow[ur, dotted] & Y. \end{tikzcd} \tag{24}$$

Clearly, the left vertical arrow could be equivalently replaced by any anodyne extension. A combinatorial spectrum X is called *Kan fibrant* if the terminal morphism $X \to *$ is a Kan fibration. This is, by adjunction, equivalent to X_n being a Kan fibrant simplicial set for every $n \in \mathbb{Z}$.

The right adjoint to $\mathscr{L}$ is the functor *Sing* which takes a May spectrum $T = (T_n)$ to a combinatorial spectrum Z where $Z_n = Sing(T_n)$. (Note that Ω commutes with *Sing*, since their left adjoints Σ and $|?|$ commute.) We shall write

$$Z = Sing(T). \tag{25}$$

Lemma 11. *(1) A combinatorial spectrum of the form (25) for a May spectrum T is Kan fibrant.*

(2) For any combinatorial spectrum Z, the unit of adjunction

$$\eta : Z \to Sing(\mathscr{L}(Z)) \tag{26}$$

is a weak equivalence.

Proof. For (1), it suffices to show that the based simplicial set $\Omega^{\infty-n}Sing(T) = Sing(T_n)$ are Kan fibrant, which is well known.

For (2), since weak equivalences are defined by applying $\mathscr{L}$, it suffices to show tht the canonical morphism

$$\mathscr{L}Sing\mathscr{L}(Z) \to \mathscr{L}(Z)$$

is a weak equivalence. More generally, we claim that the counit of adjuction

$$\epsilon : \mathscr{L}Sing(T) \to T$$

is a weak equivalence for any May spectrum T. But $\mathscr{L}Sing(T)$ is the spectrification of the inclusion prespectra whose terms are $|Sing(T_n)|$, so the statement follows from the fact that the counit of adjunction

$$|Sing(T_n)| \to T_n$$

is a weak equivalence. □

Proposition 12. *For any injective morphism of combinatorial spectra $f : X \to Y$ which is a weak equivalence, there exists a (necessarily injective) morphism of combinatorial spectra $g : Y \to Z$ such that $g \circ f$ is an anodyne extension.*

Proof. By Proposition 6, $Y = Y_\alpha$ for some ordinal α where $Y_0 = X$, for a limit ordinal β,

$$Y_\beta = \bigcup_{\gamma < \beta} Y_\gamma,$$

and for every $\beta < \alpha$, $X_{\beta+1}$ is obtained from X_β by attaching a cell of the form $\Sigma^{\infty-\ell}\Delta_{n+}$ by $\Sigma^{\infty-\ell}\Delta^\circ_{n+}$. We will construct, by induction, inclusions of spectra

$$Y_\beta \subseteq X_\beta \tag{27}$$

such that the inclusion $X \subseteq X_\beta$ is an anodyne extension. We just take unions at limit ordinals, and $X_0 = X$, so it suffices to show how to construct $X_{\beta+1}$ from X_β.

To this end, first, note that we can assume that X_β is Kan fibrant by the "small object argument" (attaching all possible $\Sigma^{\infty-\ell}\Delta_{n+}$'s via different $\Sigma^{\infty-\ell}V_{n,k+}$'s in ω steps). Now consider the attaching map

$$f : \Sigma^{\infty-\ell}\Delta^\circ_{n+} \to X,$$

and its adjoint

$$\phi : \Delta^\circ_{n+} \to X_\ell.$$

Since $(X_\beta)_\ell$ is a Kan fibrant simplicial set, and

$$X_\ell \subseteq Y_\ell$$

is a weak equivalence, ϕ extends to a morphism of based simplicial sets

$$\Delta_{n+} \to (X_\beta)_\ell,$$

and hence f extends to a morphism of Kan spectra

$$\Sigma^{\infty-\ell}\Delta_{n+} \to X_\beta.$$

In other words, "the cell we are trying to attach was already in X_β". Thus, instead of the cell, we can attach

$$\Sigma^{\infty-\ell}(\Delta_n \times I)_+$$

via

$$\Sigma^{\infty-\ell}(\Delta^\circ_n \times I \cup \Delta_n \times \{0\})_+,$$

which is an anodyne extension. □

Theorem 13. *Kan fibrant combinatorial spectra are local with respect to weak equivalences in $h\mathscr{S}$. In other words, if we denote $[X,Y] = h\mathscr{S}(X,Y)$, then for a weak equivalence $e : X \to Y$ and a Kan fibrant spectrum Z,*

$$[e,Z] : [Y,Z] \to [X,Z] \tag{28}$$

is a bijection. (Note: it is already known, and it also follows from this theorem and Lemma 11 that the weak homotopy category of combinatorial spectra is the stable category, and that $[X,Z]$ calculates morphisms in the weak homotopy category for Z Kan fibrant. Therefore, in particualar, (28) is actually an isomorphism of abelian groups.)

Proof. To prove that (28) is onto, first note that without loss of generality, we may assume that e is injective. To this end, consider the mapping cylinder

$$Me = I_+ \star X \cup_{\{1\}_+ \star X} Y. \tag{29}$$

Then by Lemma 8,

$$X = \{0\}_+ \star X \to Me. \tag{30}$$

is an inclusion, while the projection

$$Me \to \{1\}_+ \star X \cup_{\{1\}_+ \star X} Y = Y$$

is a homotopy equivalence by Lemma 9.

In effect, consider a cell decomposition of (29) considered as a relative combinatorial cell spectrum with respect to the subspectrum Y. Then for a cell a of (29), we claim that the lowest j such that for all $i > j$, $d_i(a) = *$ is the same for a considered as a cell in $I_+ \star X$. This is because whenever $d_i a \in Y$, there exists an $i' > i$ with $d_{i'} a \notin Y$, such that in $I_+ \star X$, $d_i a$ and $d_{i'} a$ have the same projection to X. (This statement depends on the fact that $I_+ \star X$ is attached to Y at the 1 coordinate.)

Therefore, by the description of $I_+ \star ?$ given below Lemma 8, $I_+ \star ?$ commutes with the pushout (29), and therefore we can apply Lemma 9.

Thus, e may be replaced with the injective morphism (30).

Now when e is injective, it is contained in an anodyne extension by Proposition 12, to which e can be extended by the assumption that Z is Kan fibrant.

To prove that (28) is injective, again, without loss of generality, we may assume that e is injective. Now form the homotopy pushout

$$Pe = Y \cup_{\{1\}_+ \star X} II_+ \star X \cup_{\{1'\}_+ \star X} Y.$$

Here II is the simplicial set obtained by attaching two copies of I by 0; the two other vertices are denoted by $1, 1'$. We have an inclusion

$$\phi : Pe \subseteq II_+ \star Y,$$

which is moreover an equivalence since e was an equivalence. Now two morphisms $f, g : Y \to Z$ which are homotopic when composed with e are the same thing as a morphism $\psi : Pe \to Z$. Since ϕ is contained in an anodyne extension by Proposition 12, ψ extends to $II_+ \star Y$, which means that f and g are homotopic, which is what we wanted to prove. □

Corollary 14. *The category of combinatorial spectra is right Cartan-Eilenberg in the sense of* [12] *with respect to homotopy equivalences, weak equivalences and Kan fibrant combinatorial spectra.*

3. Cosimplicial realization

Consider the category Δ-$\mathscr{C}$ of cosimplicial objects in $\mathscr{C}$ where $\mathscr{C}$ is either the category Δ^{Op}-$Set_\bullet$ of based simplicial sets or the category $\mathscr{S}$ of combinatorial spectra. We shall construct geometric realization functors

$$|?| : \Delta\text{-}\mathscr{C} \to \mathscr{C}. \tag{31}$$

First, let us consider the case of $\mathscr{C} = \Delta^{Op}$-$Set_\bullet$. For the terms of a cosimplicial based simplicial set X, we will use the notation X_n^m where m is the cosimplicial and n is the simplicial coordinate. We define $|X|$ as the equalizer in the category of the diagram

$$\prod_{m \in \mathbb{N}_0} F(\Delta_{m+}, X_?^m) \rightrightarrows \prod_{\phi \in \Delta^{in}(k,m)} F(\Delta_{m+}, X_?^k) \tag{32}$$

where Δ^{in} denotes the subcategory of Δ consisting of injective morphisms only, and the two arrows (32) correspond to applying the morphism in $\Delta^{in}(k,m)$ either to Δ_{m+} or to $X_?^k$.

Notice that we are "ignoring the degeneracies" in (32). Clearly, this realization of cosimplicial based simplicial sets is a functor. By a *weak equivalence* of cosimplicial based simplicial sets $f : X \to Y$ we mean a morphism such that for every m, $f_?^m : X_?^m \to Y_?^M$ is a weak equivalence. A cosimplicial based simplicial set X is *term-wise Kan fibrant* if each $X_?^m$ is Kan fibrant.

Lemma 15. *The functor* $|?|$ *preserves weak equivalences on levelwise Kan fibrant based simplicial sets.*

Proof. This follows from the fact that the canonical morphism $X \to F(\Delta_{m,+}, X)$ is a weak equivalence when X is Kan fibrant, that $F(?, X)$ turns Kan cofibrations into Kan fibrations, that pullbacks of simplicial sets along fibrations preserve weak equivalences, as do directed (inverse) limits of fibrations. □

Comment: In [2], both co-faces and codegeneracies are used in defining the cosimplicial realization of simplicial sets. While that construction seems more natural, it only preserves weak equivalences on cosimplicial simplicial sets which are fibrant in a stronger sense. Basically, one must require that the morphism from a given cosimplicial stage to the pullback of all codegeneracies from the lower stages is a fibration. We do not know if this has been checked rigorously for the case of Godement resolutions.

The difficulty is (Eckmann-Hilton) dual to a similar difficulty with the totalization (=geometric realization) of simplicial spaces. There, it is important that both faces and degeneracies be used in the realization, since only then does one have Milnor's theorem on preserving products (at least as long as we are in the compactly generated category), which, in turn, is needed when discussing algebraic structures (for example, when using the iterated bar construction to construct Eilenberg-MacLane spaces).

In the case of simplicial spaces, one may use Lillig's theorem [18] to conclude that individual degeneracies being cofibrations is enough to control the homotopy type of the totalization. As far as we know, a dual of Lillig's theorem for cosimplicial simplicial sets is not known. Nor is this, however, as urgent a problem as in the case of simplicial spaces: while in the present paper the emphasis is not on further algebraic structures, cosimplicial realization, even without co-degeneracies, preserves limits (in particular, products) by the commutation of limits.

We now apply the definition (32) to the case where X is a cosimplicial combinatorial spectrum, thus giving a definition of (31) to $\mathscr{C} = \mathscr{S}$. First, observe that a morphism of combinatorial spectra $f : Z \to T$ is a Kan fibration if and only if each $f_n : Z_n \to T_n$ is a Kan fibration. Also note that Ω preserves Kan fibrations. Finally, directed colimits also preserve Kan fibrations of simplicial sets (and hence combinatorial spectra).

4. Sheaves

We begin with sheaves of sets. We follow [24] as a reference here. A *site* is a category $\mathscr{C}$ together with a class of sets of morphisms with the same target (called *coverings*) which satisfy the usual axioms (an isomorphism is a covering, coverings are transitive, and stable under pullback), see [24], Section 6. A *presheaf valued in a category* $\mathscr{A}$ is a functor $\mathscr{F} : \mathscr{C}^{Op} \to \mathscr{A}$. The images of objects (resp. morphisms) under $\mathscr{F}$ are caled *sections* (resp. *restrictions*). A *sheaf* is a presheaf $\mathscr{F}$ such that for every covering $\{X_i \to X\}$, the diagram

$$\mathscr{F}(X) \to \prod_i \mathscr{F}(X_i) \rightrightarrows \prod_{i,j} \mathscr{F}(X_i \times_X X_j)$$

where the maps are restrictions is an equalizer. A site is said to have *subcanonical topology* it all representable presheaves are sheaves. The representable (pre)sheaf associated with an object x of a site will be denoted by $\underline{x}$.

Morphisms of presheaves are natural transformations, and sheaves are a full subcategory. The categories of $\mathscr{A}$-valued presheaves and sheaves on a site $\mathscr{C}$ will be denoted by $pSh_{\mathscr{A}}\mathscr{C}$ resp. $Sh_{\mathscr{A}}\mathscr{C}$. If the subscript is omitted, we understand $\mathscr{A} = Set$. The category of sheaves of sets on a site $\mathscr{C}$ is called its *topos*. If $\mathscr{A}$ is a category of universal algebras, then the category of $\mathscr{A}$-valued sheaves is equivalent to the category of the same type of universal algebras in the topos. In this case, the forgetful functor from sheaves valued in $\mathscr{A}$ to presheaves valued in $\mathscr{A}$ has a left adjoint called *sheafification* ([24], Section 10).

A *morphism of topoi* $f : Sh(\mathscr{C}) \to Sh(\mathscr{D})$ consists of a functor

$$f^{-1} : Sh(\mathscr{D}) \to Sh(\mathscr{C})$$

which has a right adjoint f_*, and is left exact, i.e preserves finite limits. A *point* is a morphism of a topos into the topos of sets (i.e. the category of sheaves on $*$). The set $f^{-1}\mathscr{F}$ for a sheaf $\mathscr{F}$ where f is a point is called a *stalk*. Points in this sense can be also characterized in terms of the site $\mathscr{C}$ directly ([24], Lemma 31.7). We say that a site has *enough points* if a morphism of sheaves is an isomorphism whenever it is an isomorphism on stalks. In this paper, we will work with sites $\mathscr{C}$ satisfying the following assumption:

(A1) The site $\mathscr{C}$ is small (i.e. is a set) and has enough points.

We call a morphism of sheaves *injective* resp. *surjective (or onto)* if it is injective resp. surjective on stalks. Being injective is equivalent to being injective on sections. By [24], Lemma 28.5, we may make without loss of generality (i.e. by replacing the topos with an isomorphic topos) the following assumption:

(A2) The site $\mathscr{C}$ has subcanonical topology, and subsheaves of representable sheaves are representable.

Since points are characterized in terms of the topos, in particular, we may attain Assumption (A1) without violating Assumption (A2).

Finally, Lemma 28.5 of [24] says that we can replace a (small) site with a given set of sheaves S by a (small) site satisfying Assumption (A2) where every element of S becomes representable. Choosing S to be the set of all quotients (i.e. images of surjective morphisms) of representable sheaves, we may assume $\mathscr{C}$ additionally satisfies the following assumption:

(A3) For every sheaf $\mathscr{G}$ on $\mathscr{C}$, every point p and every element $t \in p^{-1}(\mathscr{G})$, there exists an object u of $\mathscr{C}$ and an injective morphism $\underline{u} \to \mathscr{G}$ such that p is in u and t lifts to $\mathscr{G}(u)$.

Lemma 16. *Consider a site $\mathscr{C}$ satisfying Assumptions (A1) and (A2). Suppose we have injective morphisms of sheaves*

$$\alpha : \underline{x} \to \mathscr{G},\ \beta : \mathscr{F} \to \mathscr{G} \tag{33}$$

for some $x \in Obj\mathscr{C}$. Then there exists a monomorphism $\iota : y \to x$ in $\mathscr{C}$ and a pushout diagram

$$\begin{tikzcd} \underline{y} \arrow[r, "\underline{\iota}"] \arrow[d, "\phi"'] & \underline{x} \arrow[d, "\alpha"] \\ \mathscr{F} \arrow[r, "\beta"'] & \mathscr{F}' \end{tikzcd} \tag{34}$$

such that the induced morphism

$$\beta' : \mathscr{F}' \to \mathscr{G}$$

is injective.

Proof. We define the diagram (34) as the pullback of α and β. The pullback sheaf is representable by Assumption (A2). The top row is representable, and ι is a monomorphism by the Yoneda lemma. Finally, the reason β' is also injective is that it is true after applying f^{-1} (which is an exact functor) for any point f, which is enough by Assumption (A1). $\square$

To discuss homotopy theory, we begin with sheaves of simplicial sets (simplicial sheaves). Simplicial sets are a universal algebra, so the category is determined by the topos. By a *local equivalence* of simplicial sheaves (i.e. objects of $\Delta^{Op}\text{-}Sh(\mathscr{C})$), we mean a morphism which is a weak equivalence on stalks.

For a monomorphism $\iota : y \to x$ in $\mathscr{C}$, denote by Δ_n^ι the pushout in the category of simplicial sheaves

$$\begin{tikzcd} \Delta_n^\circ \times \underline{y} \arrow[r, "\Delta_n^\circ \times \iota"] \arrow[d, "\subseteq \times Id"'] & \Delta_n^\circ \times \underline{x} \arrow[d] \\ \Delta_n \times \underline{y} \arrow[r] & \Delta_n^\iota . \end{tikzcd} \tag{35}$$

Here the product of a (simplicial) set with a sheaf of sets is done section-wise. (It also commutes with taking stalks.)

An injective morphism of sheaves $f : \mathscr{F} \to \mathscr{G}$ is called a *relative cell sheaf* if we have sheaves $\mathscr{G}_\gamma$ for ordinals $\gamma < \alpha$, $\mathscr{G}_0 = \mathscr{F}$, $\mathscr{G}_\alpha = \mathscr{G}$, for a limit ordinal β, $\mathscr{G}_\beta$ is the colimit of $\mathscr{G}_\gamma$, $\gamma < \beta$, and for any ordinal $\beta < \alpha$, we have a monomorphism $\iota_\beta : y_\beta \to x_\beta$ in $\mathscr{C}$ and a pushout of sheaves of the form

$$\begin{tikzcd} \Delta_{n_\beta}^{\iota_\beta} \arrow[r] \arrow[d] & \mathscr{G}_\beta \arrow[d] \\ \Delta_{n_\beta} \times \underline{x}_\beta \arrow[r] & \mathscr{G}_{\beta+1} . \end{tikzcd} \tag{36}$$

An *anodyne extension* of sheaves is defined the same way, except $\Delta_{n_\beta}^{\iota_\beta}$ is replaced by $E_{n_\beta,k_\beta}^{\iota_\beta}$, which is defined by the pushout diagram

$$\begin{tikzcd} V_{n,k}^\circ \times \underline{y} \arrow[r, "\Delta_n^\circ \times \iota"] \arrow[d, "\subseteq \times Id"'] & \Delta_n^\circ \times \underline{x} \arrow[d] \\ \Delta_n \times \underline{y} \arrow[r] & E_{n,k}^\iota . \end{tikzcd}$$

Lemma 17. *Every injective morphism of simplicial sheaves over a site $\mathscr{C}$ which satisfies Assumptions (A1), (A2) and (A3) is a relative cell sheaf.*

Proof. Consider an injective morphism of sheaves of simplicial sets $\mathscr{F} \to \mathscr{G}$. Put $\mathscr{G}_0 = \mathscr{F}$. We will construct, inductively, sheaves $\mathscr{G}_\beta$ as in the definition of a relative cell sheaf. For β a limit ordinal, just take the colimit of $\mathscr{G}_\gamma$ over $\gamma < \beta$. For any ordinal β, we will have an injective morphism $\mathscr{G}_\beta \to \mathscr{G}$. If it is onto, we are done. Otherwise, by Assumption (A1), there exists a point p and an element $t \in p^{-1}(\mathscr{G})$

which is not in $\mathscr{G}_\beta$, but whose faces are. By Assumption (A3), there exists an object u containing p and an injective morphism $\alpha : \underline{u} \to \mathscr{G}$ such that t lifts to $\mathscr{G}(u)$ via α.

Now we are in the situation of Lemma 16, with $\mathscr{F}$ replaced by the appropriate term of $\mathscr{G}_\beta$. Let $\mathscr{G}_{\beta+1}$ be attaching one non-degenerate simplex in the appropriate dimension by the pushout (34).

The process is guaranteed to end by the smallness of $\mathscr{C}$. □

The following statement follows essentially by definition, if we perform the construction on the presheaf level first, and then sheafify.

Lemma 18. *An anodyne extension of sheaves over a site $\mathscr{C}$ which satisfies Assumptions (A1) and (A2) is a local equivalence.*

□

Lemma 19. *For an injective local equivalence of sheaves $f : \mathscr{F} \to \mathscr{G}$ over a site $\mathscr{C}$ which satisfies Assumptions (A1), (A2) and (A3), there exists a morphism of sheaves $g : \mathscr{G} \to \mathscr{H}$ such that gf is an anodyne extension.*

Proof. We shall imitate the proof of Lemma 17, expressing f as a relative cell sheaf. Using the notation in the definition, we will consruct, by induction, morphisms

$$g_\beta : \mathscr{G}_\beta \to \mathscr{H}_\beta, \tag{37}$$

$$\mathscr{G}_\beta \subseteq \mathscr{G}, \tag{38}$$

such that $g_\beta f$ is an anodyne extension. For β a limit ordinal, we can just take the directed direct limit, so let us assume (37) was constructed for a given β. Again, if (38) is onto, we are done. Otherwise, by Assumption (A1), there exists a point p and an element $t \in p^{-1}(\mathscr{G})$ which is not in $\mathscr{G}_\beta$ but whose faces are. By Assumption (A3), there exists an object u containing p and an injective morphism $\alpha : \underline{u} \to \mathscr{G}$ such that t lifts to $\widetilde{t} \in \mathscr{G}(u)$ via α and whose faces are in $\mathscr{G}_\beta$. Furthermore, since f is a local equivalence, we may assume, upon replacing u with another object $x_\beta \subseteq u$, that $\widetilde{t}$ lifts to the geometric realization of $\mathscr{H}_\beta(x_\beta)$ up to homotopy. Now considering the subobject y_β as in Lemma 16, we can assume without loss of generality that the restriction

$$\mathscr{H}_\beta(x_\beta) \to \mathscr{H}_\beta(y_\beta) \tag{39}$$

is a Kan fibration. To this end, attach, in ω steps, each time all possible pushouts of the form

$$\begin{tikzcd} E^{\iota_\beta}_{n,k} \arrow[r] \arrow[d, "\subset"'] & ? \\ \Delta_n \times \underline{x}_\beta & \end{tikzcd}$$

Despite the fact that this is not exactly the same process as the canonical factorization into an anodyne extension and Kan fibration of simplicial sets (since sheafification is performed at each time), the small object nevertheless applies, so taking the colimit over the ω steps replaces $\mathscr{G}_\beta$ with a sheaf where (39) is a Kan fibration.

Now consider the pushout defining $\mathscr{G}_{\beta+1}$ in (36). By our assumption, the composition

$$\Delta^{\iota_\beta}_{n_\beta} \to \mathscr{G}_\beta \to \mathscr{G}$$

extends to

$$\Delta_{n_\beta} \times \underline{x}_\beta \to \mathscr{G},$$

and moreover by our assumptions,

$$\Delta^{\iota_\beta}_{n_\beta} \to \mathscr{G}_\beta \to \mathscr{H}_\beta$$

extends to

$$\Delta_{n_\beta} \times \underline{x}_\beta \to \mathscr{H}_\beta \tag{40}$$

by adjunction and the assumption that (39) is a Kan fibration. (Caution: We do not know that $\mathscr{F}(x) \to \mathscr{H}_\beta(x)$ is a weak equivalence! However, it does not matter, since by assumption, the homotopy lifting problem can be solved in $|\mathscr{F}(x)|$, and in a Kan fibration, a homotopy lifting problem which has a solution upon geometric realization has a solution.)

But since we have (40), we may extend the pushout (36) to a pushout of the form

$$\begin{tikzcd} Q^{\iota_\beta}_{n_\beta} \arrow[r] \arrow[d] & \mathscr{H}_\beta \arrow[d] \\ \Delta_{n_\beta} \times I \times \underline{x}_\beta \arrow[r] & \mathscr{H}_{\beta+1} \end{tikzcd} \tag{41}$$

where Q^ι_n for a monomorphism $\iota : y \to x$ in $\mathscr{C}$ is defined as a pushout

$$\begin{tikzcd} G_n \times \underline{y} \arrow[r] \arrow[d] & G_n \times \underline{x} \arrow[d] \\ \Delta_n \times I \times \underline{y} \arrow[r] & Q^\iota_n \end{tikzcd}$$

where

$$G_n = \Delta_n^\circ \times I \cup \Delta_n \times \{0\}.$$

But (41) is an anodyne extension. Again, the process must eventually terminate for set-theoretical reasons. □

A *strong homotopy* of simplicial sheaves on $\mathscr{C}$ is a morphism of the form

$$h : \mathscr{F} \times I \to \mathscr{G}.$$

Multiplication by I is performed section-wise, and we sheafify the result. h is also called a strong homotopy between the restriction of h to $\mathscr{F} \times \{0\}$ and $\mathscr{F} \times \{1\}$. We may now consider the smallest equivalence relation on morphisms of simplicial sheaves which contains strong homotopy. This is obviously a congruence, and the quotient category is called the *strong homotopy category of simplicial sheaves.*

Lemma 20. *A strong homotopy equivalence of sheaves of simplicial sets is an equivalence on sections, and hence a local equivalence.*

Proof. For a strong homotopy of simplicial sheaves

$$I \times \mathscr{F} \to \mathscr{G}$$

and every $u \in Obj(\mathscr{C})$, we obtain, by definition, a simplicial homotopy on sections

$$I \times \mathscr{F}(u) \to \mathscr{G}(u).$$

Therefore, a strong homotopy equivalence of sheaves gives a simplicial homotopy equivalence after applying sections, hence a global equivalence. □

Theorem 21. *Under the Assumptions (A1), (A2) and (A3), and assuming also that $\mathscr{C}$ has finite cohomological dimension, the category Δ^{Op}-$Sh(\mathscr{C})$ of simplicial sheaves on $\mathscr{C}$ is right Cartan-Eilenberg with respect to strong homotopy equivalence, local equivalence and cosimplicial Godement resolutions.*

Proof. First note the following:

(42) Cosimplicial Godement resolutions have the property that restrictions under monomorphisms in $\mathscr{C}$ are Kan fibrations.

Thus, it suffices to show that for a local equivalence $e : \mathscr{F} \to \mathscr{G}$ and a cosimplicial Godement resolution $\mathscr{X}$, we have a bijection

$$[e, \mathscr{X}] : [\mathscr{G}, \mathscr{X}] \to [\mathscr{F}, \mathscr{X}] \tag{43}$$

where $[?,?]$ denotes the set of strong homotopy classes.

This is done in the standard way: To prove that (43) is onto, we first replace $\mathscr{G}$ by the mapping cylinder of e. The Lemma 19 applies (with $\mathscr{K} = \mathscr{G} \times I$). Therefore, the mapping cylinder embeds to an anodyne extension, for which the mapping extension problem into $\mathscr{X}$ can be solved by (42).

To prove that (43) is injective, first replace e by its mapping cylinder (which is isomorphic to $\mathscr{G}$ in the strong homotopy category) to make e injective. Then build a pushout $\mathscr{P}$ of two copies of e. Then the embedding

$$\mathscr{P} \subseteq \mathscr{G} \times II \tag{44}$$

is a local equivalence (where II is the simplicial set obtained from attaching two copies of I at a point) and hence it is contained in an anodyne extension. Therefore, we may conclude that for a cosimplicial Godement resolution $\mathscr{X}$, a morphism $\mathscr{P} \to \mathscr{X}$ extends to $\mathscr{G} \times II$, which shows that (43) is injective. □

The case of sheaves of based simplicial sets and combinatorial spectra is now treated analogously, with $I\times ?$ replaced by $I_+ \wedge ?$, resp. $I_+ \star ?$. Let us discuss the case of combinatorial spectra in more detail. A sheaf of combinatorial spectra is a functor from a site $\mathscr{C}$ into the category of combinatorial spectra which satisfies the sheaf limit condition in the category of combinatorial spectra. Since combinatorial spectra are a coreflexive subcategory of Δ_{st}^{Op}-$Set_\bullet$ (which is a category of universal algebras), sheafification of a presheaf of combinatorial spectra can be constructed by sheafifying the corresponding presheaf valued in Δ_{st}^{Op}-$Set_\bullet$. Since the condition of being a combinatorial spectrum cannot be called a universal algebra condition, we do not know if the category of sheaves of combinatorial spectra is independent of the choice of sites defining the same topos. Nevertheless, we shall assume that our site $\mathscr{C}$ satisfies Assumptions (A1), (A2) and (A3). By a *local equivalence* of sheaves of combinatorial spectra, we mean a morphism which is a weak equivalence on stalks.

An injective morphism of sheaves of combinatorial spectra $f : \mathscr{F} \to \mathscr{G}$ is called a *relative cell sheaf* if we have sheaves $\mathscr{G}_\gamma$ for ordinals $\gamma < \alpha$, $\mathscr{G}_0 = \mathscr{F}$, $\mathscr{G}_\alpha = \mathscr{G}$, for a limit ordinal β, $\mathscr{G}_\beta$ is the colimit of $\mathscr{G}_\gamma$, $\gamma < \beta$, and for any ordinal $\beta < \alpha$, we have an injective morphism $\iota_\beta : y_\beta \to x_\beta$

in $\mathscr{C}$ and a pushout of sheaves of the form

$$\begin{tikzcd} \Sigma^{\infty-\ell_\beta}\Delta^{\iota_\beta}_{n_\beta+} \arrow[r] \arrow[d] & \mathscr{G}_\beta \arrow[d] \\ \Sigma^{\infty-\ell_\beta}(\Delta_{n_\beta}\times \underline{x}_\beta)_+ \arrow[r] & \mathscr{G}_{\beta+1}. \end{tikzcd} \tag{45}$$

Here $\Sigma^{\infty-\ell}$ of a sheaf of based simplicial set is constructed by taking shift suspension spectra section-wise and then sheafifying. This is again left adjoint to taking $\Omega^{\infty-\ell}$ section-wise.

An *anodyne extension* of sheaves is defined the same way, except $\Delta^{\iota_\beta}_{n_\beta}$ is replaced by $E^{\iota_\beta}_{n_\beta,k_\beta}$.

Lemma 22. *Every injective morphism of sheaves of combinatorial spectra over a site $\mathscr{C}$ which satisfies Assumptions (A1), (A2) and (A3) is a relative cell sheaf.*

Proof. Consider an injective morphism of sheaves of combinatorial spectra $\mathscr{F}\to\mathscr{G}$. Put $\mathscr{G}_0=\mathscr{F}$. We will construct, inductively, sheaves $\mathscr{G}_\beta$ as in the definition of a relative cell sheaf. For β a limit ordinal, just take the colimit of $\mathscr{G}_\gamma$ over $\gamma<\beta$. For any ordinal β, we will have an injective morphism $\mathscr{G}_\beta\to\mathscr{G}$. If it is onto, we are done. Otherwise, by Assumption (A1), there exists a point p and an element $t\in p^{-1}(\mathscr{G})$ which is not in $\mathscr{G}_\beta$ but whose boundaries are. By Assumption (A3), there exists an object u containing p and an injective morphism $\alpha:\underline{u}\to\mathscr{G}$ such that t lifts to $\mathscr{G}(u)$ via α.

Now we are in the situation of Lemma 16, with $\mathscr{F}$ replaced by the appropriate term of $\mathscr{G}_\beta$. Let $\mathscr{G}_{\beta+1}$ be obtained by attaching one non-degenerate stable simplex in the appropriate dimension by the pushout (34).

The process is guaranteed to end by the smallness of $\mathscr{C}$. □

Again, the following statement follows essentially by definition, if we perform the construction on the presheaf level first, and then sheafify.

Lemma 23. *An anodyne extension of sheaves of combinatorial spectra over a site $\mathscr{C}$ which satisfies Assumptions (A1) and (A2) is a local equivalence.*

□

Lemma 24. *For an injective weak equivalence of sheaves of combinatorial spectra $f:\mathscr{F}\to\mathscr{G}$ over a site $\mathscr{C}$ which satisfies Assumptions*

(A1), (A2) and (A3), there exists a morphism of sheaves $g : \mathscr{G} \to \mathscr{H}$ such that gf is an anodyne extension.

Proof. By Lemma 22, f is a relative cell sheaf. Using the notation in the definition, we will consruct, by induction, morphisms

$$g_\beta : \mathscr{G}_\beta \to \mathscr{H}_\beta \tag{46}$$

such that $g_\beta f$ is an anodyne extension. For β a limit ordinal, we can just take the directed direct limit, so let us assume (46) has been constructed for a given β. Again, we may assume using the fact that f is a local equivalence that the embedding $\iota_\beta : y_\beta \subseteq x_\beta$ is chosen is such a way that the boundary of the new cell can be extended to the geometric realization of $\mathscr{H}_\beta$ up to homotopy. We now claim that we can assume without loss of generality that the restriction

$$\mathscr{H}_\beta(x_\beta) \to \mathscr{H}_\beta(y_\beta) \tag{47}$$

is a Kan fibration of combinatorial spectra (i.e. a level-wise Kan fibration). To this end, attach, in ω steps, each time all possible pushouts of the form

$$\begin{tikzcd}
\Sigma^{\infty-\ell} E^{\iota_\beta +}_{n,k} \arrow[r] \arrow[d, "\subset"'] & ? \\
\Sigma^{\infty-\ell}(\Delta_n \times \underline{x}_\beta)_+ &
\end{tikzcd}$$

Despite the fact that this is not exactly the same process as the canonical factorization into an anodyne extension and Kan fibration of combinatorial spectra (since sheafification is performed at each time), the small object nevertheless applies, so taking the colimit over the ω steps replaces $\mathscr{G}_\beta$ with a sheaf where (39) is a Kan fibration of combinatorial spectra.

Now consider the pushout defining $\mathscr{G}_{\beta+1}$ in (45). By our assumption, the composition

$$\Sigma^{\infty-\ell_\beta} \Delta^{\iota_\beta}_{n_\beta +} \to \mathscr{G}_\beta \to \mathscr{G}$$

extends to

$$\Sigma^{\infty-\ell_\beta}(\Delta_{n_\beta} \times \underline{x}_\beta)_+,$$

which by our assumption on ι_β means

$$\Sigma^{\infty-\ell_b eta} \Delta^{\iota_\beta}_{n_\beta +} \to \mathscr{G}_\beta \to \mathscr{H}_\beta$$

extends to

$$\Sigma^{\infty-\ell_\beta}(\Delta_{n_\beta} \times \underline{x}_\beta)_+ \to \mathscr{H}_\beta \tag{48}$$

by adjunction and the assumption that (47) is a Kan fibration of combinatorial spectra. (Caution: We do not know that $\mathscr{F}(x) \to \mathscr{H}_\beta(x)$ is a weak equivalence! However, it does not matter, since by assumption, the homotopy lifting problem can be solved in $|\mathscr{F}(x)|$, and in a Kan fibration of combinatorial spectra, just as of simplicial sets, a homotopy lifting problem which has a solution upon geometric realization has a solution.)

But since we have (48), we may extend the pushout (45) to a pushout of the form

$$\begin{tikzcd}
\Sigma^{\infty-\ell_\beta}Q^{\iota_\beta}_{n_\beta+} \arrow[r] \arrow[d] & \mathscr{H}_\beta \arrow[d] \\
\Delta_{n_\beta}\times I\times \underline{x}_\beta \arrow[r] & \mathscr{H}_{\beta+1}
\end{tikzcd} \tag{49}$$

where Q^ι_n is as in the proof of Lemma 19. □

A *strong homotopy* of sheaves of combinatorial spectra on $\mathscr{C}$ now is a morphism of the form

$$h : I_+ \star \mathscr{F} \to \mathscr{G}.$$

Again, $I_+\star?$ is performed section-wise, and we sheafify the result. h is also called a strong homotopy between the restriction of h to $\{0\}_+\star\mathscr{F}$ and $\{1\}_+\star\mathscr{F}$. We may now consider the smallest equivalence relation on morphisms of simplicial sheaves which contains strong homotopy. This is obviously a congruence, and the quotient category is called the *strong homotopy category of sheaves of combinatorial spectra.*

Lemma 25. *A strong homotopy equivalence of sheaves of combinatorial spectra is an equivalence on sections, and hence a local equivalence.*

Proof. Analogous to the proof of Lemma 20. □

Theorem 26. *Under the Assumptions (A1), (A2) and (A3), and assuming also that $\mathscr{C}$ has finite cohomological dimension, the category $Sh_{\mathscr{S}}(\mathscr{C})$ of sheaves of combinatorial spectra on $\mathscr{C}$ is right Cartan-Eilenberg with respect to strong homotopy equivalence, local equivalence and cosimplicial Godement resolutions.*

Proof. First note the following:

(50) Cosimplicial Godement resolutions have the property that restrictions under monomorphisms in $\mathscr{C}$ are Kan fibrations of combinatorial spectra.

Thus, it suffices to show that for a local equivalence of combinatorial spectra $e : \mathscr{F} \to \mathscr{G}$ and a cosimplicial Godement resolution $\mathscr{X}$, we have a bijection

$$[e, \mathscr{X}] : [\mathscr{G}, \mathscr{X}] \to [\mathscr{F}, \mathscr{X}] \tag{51}$$

where $[?, ?]$ denotes the set of strong homotopy classes.

This is done in the standard way: To prove that (51) is onto, we first replace $\mathscr{G}$ by the mapping cylinder of e. The Lemma 19 applies (with $\mathscr{K} = I_+ \star \mathscr{G}$). Therefore, the mapping cylinder embeds to an anodyne extension, for which the mapping extension problem into $\mathscr{X}$ can be solved by (50).

To prove that (51) is injective, first replace e by its mapping cylinder (which is isomorphic to $\mathscr{G}$ in the strong homotopy category) to make e injective. Then build a pushout $\mathscr{P}$ of two copies of e. We note that we do not know that the embedding

$$II_+ \star \mathscr{P} \subseteq \mathscr{G} \tag{52}$$

is a local equivalence (where II is again the simplicial set obtained from attaching two copies of I at a point). Thus, we know (52) is contained in an anodyne extension of $\mathscr{P}$. Therefore, we may still conclude that for a cosimplicial Godement resolution $\mathscr{X}$, a morphism $\mathscr{P} \to \mathscr{X}$ extends to $II_+ \star \mathscr{G}$, which shows that (51) is injective. □

5. Comparison with Thomason

Thomason [25] considers a concept of *fibrant simplicial spectra* (which he attributes to Bousfield and Friedlander [2]) which are sequences

$$Z_n, n \in \mathbb{N}_0$$

(alternatively, $n \in \mathbb{Z}$) of based simplicial sets, together with structure morphisms

$$S^1 \wedge Z_n \to Z_{n+1}, \tag{53}$$

such that the simplicial sets Z_n are Kan fibrant, and the adjoints of the structure maps (53) are weak equivalences:

$$Z_n \xrightarrow{\sim} F(S^1, Z_{n+1}).$$

Morphisms $(Z_n) \to (T_n)$, as usual, are sequences of morphisms of based simplicial sets which commute with the structure morphisms (53).

Thomason [25] observes that his category of fibrant simplicial spectra has products and directed colimits, which means that on sites with enough points, local equivalences may be defined as morphisms which induce weak equivalences on stalks. Global equivalence are defined as morphisms which induce global equivalences on sections. Additionally, on sites which have finite cohomological dimension, where one can use cosimplicial Godement resolutions, one can treat generalized sheaf cohomology completely on the level of presheaves. The reason is that stalks, again, can be calculated on the level of presheaves and on the other hand, the products of skyscaper sheaves which occur in cosimplicial Godement resolutions are sheaves in any subcanonical topology. It was proved in [23] that under these assumptions, the category of Thomason presheaves of fibrant simplicial spectra is right Cartan-Eilenberg with respect to global equivalences, local equivalences and Godement resolutions.

An actual general theory of sheaves of Thomason's fibrant fibrant simplicial spectra, on the other hand, does not seem meaningful, as more types of colimits are necessary, for example, to have sheafification.

By Proposition 1, we have a canonical natural transformation

$$S^1 \wedge X = \Sigma(S^0) \wedge X \to \Sigma(S^0 \wedge X) = \Sigma X. \tag{54}$$

This means that Kan fibrant combinatorial spectra are canonically a full subcategory of Thomason fibrant simplicial spectra, and local and global equivalences coincide. By Lemma 11 (2), every combinatorial spectrum can be functorially replaced by a Kan fibrant one, so doing this sectionwise gives a functor from sheaves of combinatorial spectra to presheaves of Kan simplicial spectra which preserves local and global equivalences.

There is also a functor the other way. In fact, the construction applies to any sequence of based simplicial sets with connecting morphisms (53) without any additional assumptions. We may then apply functorial fibrant replacement. The construction is performed in two steps. First, replace Z_n with a sequence of based simplicial sets Z'_n and connecting maps (53) with Z_n replaced by Z'_n which are injective. We define inductively

$$Z'_0 = Z_0,$$

and assuming we already have a morphism

$$Z'_n \to Z_n,$$

we let Z'_{n+1} be the mapping cylinder of the composition

$$S^1 \wedge Z'_n \to S^1 \wedge Z_n \to Z_{n+1}.$$

Thus, assume without loss of generality $Z' = Z$. In the second step, we make this into a combinatorial prespectrum as follows: Let

$$Z''_0 = Z_0.$$

Assuming we have already a morphism of based simplicial sets

$$Z_n \to Z''_n,$$

we let Z''_{n+1} be the pushout of the canonical diagram

$$\begin{tikzcd} S^1 \wedge Z_n \arrow[r] \arrow[d] & Z_{n+1} \\ \Sigma Z''_n. & \end{tikzcd}$$

By functoriality, this construction automatically passes to presheaves. We may then spectrify, sheafify and fibrant replace as we wish. All those functors are left adjoint and commute by commutation of adjoints.

The advantage of using combinatorial spectra, where we have a fully functional theory of sheaves, is that we now also have functors (and their right derived functors, provided they preserve strong homotopy) which cannot be constructed on presheaves alone. Functors of the form f_* for a general morphism of sites f are an example.

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