ON THE STRUCTURE OF EQUIVARIANT FORMAL GROUP LAWS

I.KRIZ, Y.LU

Abstract. We give a new proof of Greenlees's conjecture stating that for an abelian compact Lie group G, the stable G-equivariant complex cobordism ring MU_*^G is the G-equivariant Lazard ring, i.e. the ring representing universal G-equivariant formal group laws. In the process, we give a presentation of MU_*^G as a deformation of the non-equivariant Lazard ring $L = MU_*$.

1. Introduction

An $RO(G)$-graded equivariant generalized cohomology theory (or G-spectrum) for a compact Lie group G ([10]) is called *complex-oriented* if it has a Thom isomorphism with respect to every G-equivariant complex bundle. In the case of G abelian, Cole, Greenlees and Kriz [**?**, 2] introduced a concept of a *G-equivariant formal group law*, which is the algebraic structure universally present on the cohomology ring $R = E^{even}(\mathbb{C}P_G^\infty)$ for a complex-oriented G-equivariant spectrum E where $\mathbb{C}P_G^\infty$ is the equivariant infinite-dimensional complex projective space. Roughly speaking, the structure is a formal group scheme (not necessarily Noetherian) over $Spec(A)$ where $A = E_{even}$ which receives a morphism of formal group schemes from the formal group of characters $\widehat{G}$ of G, which has a "good parameter," i.e. a regular element x such that $R/(x) = A$ and a fundamental system of neighborhoods in R is given by the ideals generated by products of $\widehat{G}$-images of x. We shall give a precise definition in Section 2.

It was conjectured by J.P.C.Greenlees that the ring A which supports the universal G-equivariant formal group law for an abelian compact Lie group G is isomorphic to the stable equivariant cobordism ring MU_*^G [14, 10]. This conjecture was recently proved first for $G = \mathbb{Z}/2$ by Hanke and Wiemeler [5], and then in full generality by M. Hausmann [6]. The purpose of this paper is to give a different proof of Hausmann's theorem [6], which also comes with a different statement, and gives a certain insight into the structure of the equivariant Lazard

Kriz acknowledges the support of a Simons Collaboration Grant.

ring MU^G_*. The structural information we obtain is somewhat different from the theorems of P. Hu [7], who completely describes the divisibility properties of the cobordism ring with respect to Euler classes in the case of $G = \mathbb{Z}/p^r$. Instead, we obtain a formulation which, in some sense, presents the G-equivariant Lazard ring as a deformation of the non-equivariant Lazard ring.

We will need more preliminaries to be completely precise, but briefly, we can describe our statement as follows. A *G-complete flag E* can be identified with a sequence of irreducible (hence 1-dimensional) complex representations $\alpha_1, \alpha_2, \dots\dots$ of G which contains every irreducible representation infinitely many times. For a G-equivariant formal group law (A, R), with a complete flag E, we can associate elements

$$x_{E,n} = \prod_{i=1}^{n} x_{\alpha_i}$$

where x_α is the translate of the good parameter x by the character α. Additionally, as an A-module, R is a product of copies of A, indexed by the generators $x_{E,n}$, $n = 0, 1, 2, \dots$. The comultiplication then gives

$$x \mapsto \sum_{i,j \geq 0} a^E_{ij} x_{i,E} \otimes x_{j,E}. \tag{1}$$

Additionally, we have Euler classes u_α which are reductions of $x_{\alpha^{-1}}$ in $R/(x) = A$. Now by completing at (x), we can express the right hand side of (1) as a power series in $1 \otimes x$ and $x \otimes 1$ with coefficients which are polynomials in the Euler classes with coefficients in A. In this sense, the coefficients a_{ij} of the underlying non-equivariant formal group law

$$x \mapsto \sum_{i,j} a_{ij} x^i \otimes x^j$$

can be expressed as polynomials in the generators a^E_{ij} and the Euler classes, and in fact, it turns out that a_{ij} is congruent to a^E_{ij} modulo the ideal generated by the Euler classes.

Thus, the relations between coefficients of the non-equivariant formal group law lead to one set of relations between the the elements a^E_{ij} and u_α. In addition to these relations, we also have that

$$u_e = 0 \tag{2}$$

where e is the trivial irreducible representation. Because of this, (1) gives the relation

$$u_{\alpha \otimes \beta} = \sum_{i,j} a^E_{ij} (u_\alpha)_{i,E} (u_\beta)_{j,E} \tag{3}$$

where

$$(u_\alpha)_{n,E} = \prod_{i=1}^{n} u_{\alpha\otimes\alpha_i}.$$

Further, because of (2), the right hand side of (3) has only finitely many non-zero summands, and thus defines a second set of relations between the generators a_{ij}^E and u_α.

Our main result then can be stated as follows:

Theorem 1. *The G-equivariant Lazard ring (i.e. the ring A supporting the universal G-equivariant formal group law (A, R)) is the quotient of the polynomial ring*

$$\mathbb{Z}[a_{ij}^E, u_\alpha]$$

modulo the two types of relations given above. Additionally, it is isomorphic to the stable G-equivariant cobordism ring MU_*^G.

We can see that this description can be considereda deformation of the non-equivariant Lazard ring, which we get by quotienting out the Euler classes. More precise reformulations and proof of this theorem occupy the remainder of this paper.

The present paper is organized as follows: The basic notations and definitions we are using are given in Section 2. The case of a cyclic group, which is somewhat special, is treated in Section 3. The case of S^1 is proved in Section 4 and the general case is proved in Section 5.

2. Definitions and notation

In this section, we give the needed definitions and notation.

Definition 2. Let G be a finite abelian group. A *G-equivariant formal group law* $(A, R, \Delta, \epsilon, x_L)$ (sometimes abbreviated as (A, R)) consists of the following data:

(1) A commutative ring A and a commutative A-algebra R which is complete with respect to an ideal I where $I \cap A = 0$, together with a homomorphism

$$\Delta : R \to R \widehat{\otimes}_A R \tag{4}$$

(where $R\widehat{\otimes}_A R = (R \otimes_A R)^\wedge_J)$ where J is the ideal generated by $x \otimes 1, 1 \otimes x$, $x \in I$.

(2) An I-continuous map of rings $\epsilon : R \to A[\widehat{G}]^\vee$ where the target has the discrete topology, for an A-module M, $M^\vee = Hom_A(M, A)$

and $\widehat{G} = Hom(G, S^1)$ is the Pontrjagin dual, compatible with comultiplication,

$$
\begin{tikzcd}
R \arrow[r, "\Delta"] \arrow[d, "\epsilon"'] & R\widehat{\otimes}_A R \arrow[d, "\epsilon\widehat{\otimes}_A\epsilon"] \\
A[\widehat{G}]^\vee \arrow[r, "\psi"] & A[\widehat{G}]^\vee \otimes_A A[\widehat{G}]^\vee
\end{tikzcd}
\tag{5}
$$

where the comultiplication ψ

$$A[\widehat{G}]^\vee \xrightarrow{\psi} A[\widehat{G}]^\vee \otimes_A A[\widehat{G}]^\vee$$

is given by the group structure on G.

(3) A system of regular elements $x_L \in R$, $L \in \widehat{G}$ such that $R/(x_L) \cong A$ as A-algebras,

$$I = \prod_{L\in\widehat{G}} (x_L)$$

and

$$x_L = (\epsilon(L) \otimes 1)\Delta(x_1) \text{ for } L \in \widehat{G}. \tag{6}$$

The element $x_1 \in R$ is often denoted by x. Note that we can consider $Spf(R)$ as a formal group scheme over $Spec(A)$, although we do not require that $Spec(A)$ be Noetherian. From this point of view, $Spec(A[\widehat{G}])$ is just the (discrete) character group over $Spec(A)$ and the diagram (5) is a morphism of formal group schemes over $Spec(A)$. The existence of the "good parameter" x and the expression of the ideal of definition of $Spf(R)$ by condition (3) however is a substantial additional restriction which is the equivariant analogue of specializing from non-equivariant formal groups to formal group laws.

This definition generalizes to the case when G is an abelian compact Lie group with some changes. The group algebra $A[\widehat{G}]$ is discrete and its dual $A[\widehat{G}]^\vee$ is given the product topology. We do not have an ideal I and instead, the condition (1) says that the ring R is complete with respect to the topology where a fundamental system of neighborhoods of 0 is given by all the ideals

$$(x_{L_1} \cdot \dots \cdot x_{L_n})$$

for $L_1, \dots, L_n \in \widehat{G}$ and the elements x_L are still related to the "good parameter" x by (6). The ring $R \otimes_A R$ in (4) is completed with respect to the topology where a fundamental system of neighborhoods of 0 is given by the ideals

$$(x_{L_1} \cdot \dots \cdot x_{L_n} \otimes 1, 1 \otimes x_{L_1} \cdot \dots \cdot x_{L_n})$$

where $L_1, \dots, L_n \in \widehat{G}$. Thus, in the case of an infinite abelian compact Lie group G, we are dealing with a further generalization of the concept of a formal group scheme in that R does not really have an "ideal of definition."

For a compact Lie group G, a *G-equivariant May spectrum E* [10] is a representing object for a homology and cohomology theory $E^\alpha(X)$ on G-spaces X, $\alpha \in RO(G)$ where $RO(G)$ is the real representation ring of G. We have a suspension axiom on reduced (co)homology of based G-CW-complexes. (The usual discussion on the connection between reduced and unreduce (co)homology applies.) A commutative associative G-equivariant ring spectrum E is defined analogously as in the non-equivariant case by a product morphism $\mu : E \wedge E \to E$ and a unit morphism $\eta : S \to E$ satisfying analogous axioms as in the non-equivariant case. Then for every closed subgroup $H \subseteq G$, $\pi_*(E^H)$ forms a graded-commutative ring. We refer the reader to [10] for details.

A commutative associative G-equivariant ring spectrum E is called *complex-oriented* if for every G-equivaraint n-dimensional complex bundle ξ on a G-CW complex X there exists a *Thom class* $u \in \widetilde{E}^{2n}X^\xi$ (where X^ξ denotes the Thom space) which restricts to a unit in $\pi_0 E^H$ on every fiber of X^H. Plugging in u to the Thom diagonal $\theta : X^\xi \to X^\xi \wedge X_+$ then gives a Thom isomorphism

$$E^\alpha X \cong \widetilde{E}^{\alpha+2n} X^\xi.$$

As a consequence, one gets *complex stability* which is a natural isomorphism

$$E^\alpha X \cong E^{\alpha+\kappa} X$$

for every virtual representation κ of dimension 0.

The G-equivariant infinite complex projective space $\mathbb{C}P^\infty_G$ can be defined as the G-space of all complex lines in the *complete uiverse* $\mathcal{U}$, which is a disjoint sum of countably many copies of all irreducible representations of G. It was proved in [**?**] that for an abelian compact Lie group G and a G-equivariant complex-oriented spectrum E, there is a natural G-equivariant formal group law (A, R) where $A = E_{even}$ (the commutative ring of $2\mathbb{Z}$-graded (co)homology groups of a point), $R = E^{even}\mathbb{C}P^\infty_G$. The comultiplication on R is induced by the multiplication on $\mathbb{C}P^\infty_G$ coming from the tensor product.

A particular example of a complex-oriented G-equivariant spectrum (see [2]) is the Thom spectrum MU_G which can be defined as the colimit of the Thom spectra $BU_G(n)^{\gamma^n - 2n}$ where $BU_G(n)$ can be defined as the Grassmannian of n-dimensional subspaces of $\mathcal{U}$ and γ^n is the universal G-equivariant complex n-bundle, whose fiber over $V \in BU_G(n)$ is the space V. It was conjectured by Greenlees that for G abelian compact Lie, the G-equivariant formal group law of MU_G is the universal G-equivariant formal group law. In particular, the conjecture was that $(MU_G)_{even} = L_G$ is the *G-equivariant Lazard ring*, i.e. the representing object of G-equivariant formal group laws. One can prove directly that there exists such a ring L_G for which there is a natural bijection between ring homomorphisms $L_G \to A$ and G-equivariant formal group laws (A, R).

The Greenlees conjecture was proved first for $G = \mathbb{Z}/2$ by Hanke and Wiemeler [5] and more recently in the general case by Hausmann [6]. Theorem 1 above contains that result, giving also in some sense an explicit description of the equivariant Lazard ring $(MU_G)_{even}$. As it turns out, this ring is concentrated in even degrees by the theorem of Comezana and May [3], which will also follow from our proof. We will therefore denote this ring simply by $(MU_G)_*$ or MU^G_*.

3. The case of $G = \mathbb{Z}/n$

In this section, we will discuss Theorem 1 in the case of a cyclic group $G = \mathbb{Z}/n$. In fact, we will first discuss a particularly nice G-complete flag in this case, for which the theorem has a simpler statement, which is easier to prove.

We use the symbol 1 for the trivial representation, and α for rotation by $2\pi/n$. Define the *regular* flag $\{V_i\}_{i\geq 0}$ by

$$V_0 = 0,\ V_1 = 1,\ V_2 = 1+\alpha,\ V_3 = 1+\alpha+\alpha^2 \dots \tag{7}$$

With the complete flag the ring R has an additive basis $\{x_i : i \geq 0\}$, where $x_i = x_{V_i}$ is the product of the coordinates corresponding to components of V_i. Given any $\mathbb{Z}/n$-equiavariant formal group law we can hence under the basis write the coproduct of x as

$$\Delta(x) = \sum_{i,j\geq 0} \overline{a_{i,j}}\, x_i \otimes x_j. \tag{8}$$

The structure constants $\overline{a_{i,j}} \in A$ will satisfy certain relations from the axioms of equivariant formal group laws. For example, the cocommutativity condition implies that $\overline{a_{i,j}} = \overline{a_{j,i}}$, and the counitality condition implies that $\overline{a_{i,j}} = 0$ if $i = 0$ and $j \neq 1$.

We define the *Euler class of α* by

$$u = \epsilon(\alpha^{-1})(x)$$

where $x = x_1$. More generally, we can define

$$u_k = \epsilon(\alpha^{-k})(x)$$

for $k = 1, 2, ..., n$. We also have

$$u_k = \epsilon(\alpha^{-\ell})(x_{\alpha^{\ell-k}}),$$

$$u_0 = 0.$$

Thus, inductively we can compute,

$$\begin{aligned} u_k &= \epsilon(\alpha^{-k})(x) = (\epsilon(\alpha^{-1}) \otimes \epsilon(\alpha^{1-k}))(\Delta(x)) \\ &= u_{k-1} + u \cdot (\sum_{i=0}^{k-1} \overline{a_{1,i}}(\prod_{j=0}^{i-1} u_{k-1-j})). \end{aligned} \tag{9}$$

Thus, referring to (9), we obtain a relation among the elements $u, \overline{a_{i,j}} \in A$ given by

$$u_n = 0 \tag{10}$$

which has the form

$$nu \mod (u^2). \tag{11}$$

We will also need the following result: let p be a prime factor of n, and define $m = n/p$. Similar computation gives

$$\begin{aligned} u_{km} &= \epsilon(\alpha^{-km})(x) = (\epsilon(\alpha^{-m}) \otimes \epsilon(\alpha^{m-km}))(\Delta(x)) \\ &= (\epsilon(\alpha^{-m}) \otimes \epsilon(\alpha^{m-km}))(\sum_{i,j \geq 0} \overline{a_{i,j}} x_i \otimes x_j) \\ &= u_m + u_{(k-1)m} + \sum_{i=1}^{m} \sum_{j=1}^{(k-1)m} \overline{a_{i,j}}(\prod_{s=0}^{i-1} u_{m-s})(\prod_{t=0}^{j-1} u_{(k-1)m-t}). \end{aligned} \tag{12}$$

The relation (10) therefore has an alternative form

$$pu_m \mod (u_m^2). \tag{13}$$

Another set of relations is obtained as follows. If $(A, R, \Delta, \epsilon, x_L)$ is a $\mathbb{Z}/n$-equivariant formal group law, then $R^{\wedge}_{(x)} = A[[x]]$ (since x is regular, $R/(x) = A$ implies $R/(x^n) \cong A[x]/(x^n)$). Thus, applying the completion map

$$R \to A[[x]],$$

the coproduct Δ maps to a non-equivariant formal group law on A. By Lazard's theorem [9], we obtain an expression of the coefficients $a_{i,j}$ as polynomials of $\overline{a_{i,j}}$ and u. Note also that, by the definition of $\overline{a_{i,j}}$, with this identification, we have

$$\overline{a_{i.j}} \equiv a_{i,j} \mod (u) \tag{14}$$

(since $x_L \equiv x \mod (u)$). Thus, the relations

$$r(a_{i,j}) \tag{15}$$

in the Lazard ring give, by substitution, a set of relations among the $\overline{a_{i,j}}$'s and u. (Recall that modulo decomposables, the relations among the $a_{i.j}$'s say that they are all multiples of the Lazard generators x_{i+j}, which in turn, modulo indecomposables, is a linear combination of the $a_{i,j}$'s with coefficients in $\mathbb{Z}$.)

Now we state a version of Theorem 1 for the regular flag $\{V_i\}_{i\geq 0}$:

Theorem 3. *(1) There exists a $\mathbb{Z}/n$-equivariant formal group law*

$$(A, R, \Delta, \epsilon, x_L)$$

where A is the quotient ring of $\mathbb{Z}[u, \overline{a_{i,j}}]$ modulo the relations (10), (15). Furthermore, this $\mathbb{Z}/n$-equivariant formal group law is universal in the sense that for any $\mathbb{Z}/n$-equivariant formal group law $(A', R', \Delta', \epsilon', x'_L)$, there exists unique ring homomorphisms $A \to A'$, $R \to R'$, which carries Δ to Δ', ϵ to ϵ', and x_L to x'_L.

(2) There is an isomorphism $A \cong (MU_{\mathbb{Z}/n})_*$.

The theorem will be proved by induction. For clarity we will sometimes write A_n for the ring A in the statement above. Throughout we use the complete flag $\{V_i\}_{i\geq 0}$ defined above.

Proof. First, we observe that if we have a homomorphism of rings $f : A \to A'$, there exists at most one equivariant formal group law $(A', R', \Delta', \epsilon', x'_L)$ over A' such that (8) holds with $\overline{a_{i,j}}$ replaced by $f(\overline{a_{i,j}})$, and other notations replaced. Applying f to the coefficients of the computations below (8), we get formulas for all euler classes $u'_k \in A'$ in terms of the images of $\overline{a_{i,j}}$'s. Now

$$\epsilon'(\alpha^{-1})(x_k) = \prod_{i=1}^{k} u'_i \in A'.$$

Therefore,

$$\epsilon'(\alpha^{-\ell})(x_k) = \prod_{i=1}^{k} u'_{i+\ell}.$$

This determines ϵ' by linear extension. Since

$$\Delta'(x') = \sum_{i,j} f(\overline{a_{i,j}})\, x'_i \otimes x'_j,$$

the following formula determines elements

$$x'_L = (\epsilon'(L) \otimes 1)\Delta'(x_1).$$

For example when $p = 2, L = \alpha$ this gives

$$x'_\alpha = u' + x'_1 + u' f(\overline{a_{1,1}}) x'_1 + u' f(\overline{a_{1,2}}) x'_2 + \ldots).$$

It holds a priori that

$$x'_\alpha x'_1 = x'_2,$$
$$x'_{\alpha^2} x'_2 = x'_3,$$
$$\ldots$$

To calculate $x'_{\alpha^\ell} x'_k$ for any $\ell \in \mathbb{Z}/n$ and $k \geq 1$, the recipe is to rotate the above formula for $x'_{\alpha^{\ell-k}}$ by α^k. This gives x'_{α^ℓ} in terms of a new basis

$$\{V_{k+i}/V_k\}_{i\geq 1} = \{\alpha^k V_i\}_{i\geq 1}.$$

Now since x'_k corresponds to V_k and

$$(V_{k+i}/V_k) \oplus V_k \cong V_{k+i},$$

we have $x'_{\alpha^\ell} x'_k$ in original basis $\{V_i\}_{i\geq 1}$.

Similarly, we can compute $x'_{\alpha^\ell} x'_m$ for any ℓ, m. Thus, by induction, the product in R' is determined.

Now by Axiom (2),

$$\begin{aligned}
&\Delta'(x'_{\alpha^\ell}) = \Delta'(\epsilon'(\alpha^\ell) \otimes Id) \circ \Delta'(x')) \\
&= ((\epsilon'(\alpha^\ell) \otimes Id) \circ \Delta') \otimes (\epsilon') \otimes Id) \circ \Delta') \circ \Delta'(x') \\
&= \sum_{i,j} f(\overline{a_{i,j}}) \prod_{k=1}^{i} x'_{\alpha^{\ell+k-1}} \otimes x'_j,
\end{aligned}$$

(since $(\epsilon'(\alpha^\ell) \otimes Id) \circ \Delta'(x'_i) = \prod_{k=1}^{i} x'_{\alpha^{\ell+k-1}}$). Thus Δ' is also determined.

Note that we do not yet know that A actually supports a $\mathbb{Z}/n$-equivariant formal group law. However, we have the following

Lemma 4. *The ring $A_{\mathbb{Z}/n}$ has u_m-torsion of order ≤ 1, i.e. for every $z \in A_{\mathbb{Z}/n}$, if $u_m^2 z = 0$, then $u_m z = 0$.*

Proof. It will be convenient to introduce the polynomial generators x_k of the (non-equivariant) Lazard ring L. We also join another formal generator u_m to A. Then we can write

$$A = \mathbb{Z}[\overline{a_{i,j}}, x_k, u, u_m]/(r_{i,j}, u_m = [m]u, [p]u_m)$$

where the relations

$$r_{i,j} = a_{i,j} - q_{i,j}(x_k)$$

are given by thinking of $a_{i,j}$ as a polynomial in the $\overline{a_{i',j'}}$'s and u, and

$$[p]u_m, \quad u_m = [m]u$$

are results from computations (9), and

$$[p]u_m$$

is (13) of form

$$pu_m \mod u_m^2.$$

Suppose

$$u_m^2 \mid Q = c\cdot[p]u_m + \sum c_{i,j}r_{i,j} + d\cdot[m]u$$

for some $c_{i,j}, c, d \in \mathbb{Z}[\overline{a_{i,j}}, x_k, u, u_m]$. Then, factoring out u_m and considering non-equivariant formal group law theory (namely the algebraic independence of the relations $r_{i,j}$), we conclude that

$$u_m \mid c_{i,j},\ u_m \mid d$$

for all i, j. Thus, we may write

$$\text{coeff}_{u_m}(Q) = cp + \sum \frac{c_{i,j}}{u_m} r_{i,j} + \frac{d}{u_m}. \tag{16}$$

Then assuming $u_m \nmid c$, factoring out u_m, (16) would again contradict the algebraic independence of the relations $r_{i,j}$ of the classical (non-equivariant) Lazard ring. Therefore, $u_m \mid c$, and therefore the relation Q can be divided by u_m, as claimed. □

Recall that we choose $p \mid n$ and let $m = n/p$. Let $\mathscr{F} = \mathscr{F}(\mathbb{Z}/m)$ be the family of subgroups of $\mathbb{Z}/n$ contained in $\mathbb{Z}/m$. Denote the universal space for this family by $E\mathbb{Z}/p$ (since $E\mathbb{Z}/p$ is a model for $E\mathscr{F}(\mathbb{Z}/m)$) and consider the cofiber $\widetilde{E\mathbb{Z}/p}$ of $E\mathbb{Z}/p_+ \to S^0$, we have a pullback square for equivariant complex cobordism

$$\begin{tikzcd} MU_{\mathbb{Z}/n} \arrow[r] \arrow[d] & \widetilde{E\mathbb{Z}/p} \wedge MU_{\mathbb{Z}/n} \arrow[d] \\ F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/n}) \arrow[r] & \widetilde{E\mathbb{Z}/p} \wedge F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/n}). \end{tikzcd} \tag{17}$$

On the other hand, by [11], we have a pullback square for $A = A_{\mathbb{Z}/n}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & u_m^{-1}A \arrow[d] \\ A^{\wedge}_{u_m} \arrow[r] & u_m^{-1}A^{\wedge}_{u_m}. \end{tikzcd} \tag{18}$$

By Greenlees and May [4], the coefficient of the square (17) is

$$\begin{tikzcd} (MU_{\mathbb{Z}/n})_* \arrow[r] \arrow[d] & u_m^{-1}(MU_{\mathbb{Z}/n})_* \arrow[d] \\ (MU_{\mathbb{Z}/n})^{\wedge}_{*u_m} \arrow[r] & u_m^{-1}(MU_{\mathbb{Z}/n})^{\wedge}_{*u_m}. \end{tikzcd} \tag{19}$$

Now we shall argue that the top row and the left hand column actually define $\mathbb{Z}/n$-equivariant formal group laws on the respective target rings, and the rings are isomorphic to the corresponding coefficients in the square (19).

In the case of the bottom left corner, we start with the case when n is a prime. In this case, our conclusion is due to the fact that if we are allowed to sum infinite power series in u, then the $\overline{a_{i,j}}$'s can also be expressed as power series in the $a_{i,j}$'s (rather than just vice versa). Under this correspondence, the relation (10) just becomes $[p]u = 0$. Thus,

$$A^{\wedge}_{(u)} \cong L[[u]]/[p]u$$

where L is the non-equivariant Lazard ring.

In the general case, use induction assumption we have

$$A_{\mathbb{Z}/n}/u_m \cong A_{\mathbb{Z}/m} \cong (MU_{\mathbb{Z}/m})_* \cong (MU_{\mathbb{Z}/n})_*/u_m$$

(the first isomorphism follows from our definition and the last isomorphism comes computations for equivariant complex cobordism). We can use Borel cohomology spectral sequence to compute the associated graded ring of $((MU_{\mathbb{Z}/n})_*)^{\wedge}_{u_m}$

$$E_2^{s,t} = H^s(\mathbb{Z}/p, \pi_t(MU_{\mathbb{Z}/n})) \Rightarrow \pi_* F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/n}).$$

It collapses since the coefficient concentrates in even degrees and there are no p-torsions, and gives the associated graded ring as

$$(MU_{\mathbb{Z}/m})_*[[u_m]]/[p]u_m.$$

Denote $(MU_{\mathbb{Z}/n})_*$ by S and denote u_m by ω:

$$\omega^r S/\omega^{r+1}S \cong \omega^r S^{\wedge}_{\omega}/\omega^{r+1}S^{\wedge}_{\omega} \cong S/(\omega, p).$$

Compare the short exat sequence

$$0 \to pS/\omega S \to S/\omega S \to S/(\omega, p) \cong \omega^r S/\omega^{r+1}S \to 0$$

with

$$0 \to \ker q \to A/\omega A \xrightarrow{q} \omega^r A/\omega^{r+1}A \to 0$$

and $[p]\omega/\omega \in \ker q$ maps to $p \in pS/\omega S$, hence the map

$$\ker q \to pS/\omega S$$

is surjective and

$$\omega^r A/\omega^{r+1}A \to \omega^r S/\omega^{r+1}S$$

is an isomorphism for all $r \geq 1$. Thus,

$$A^{\wedge}_{u_m} \cong ((MU_{\mathbb{Z}/n})_*)^{\wedge}_{u_m}. \tag{20}$$

Let $p_1, \ldots, p_s$ be the different prime factors of n, $p = p_1$. Similarly to (20), we can prove

$$(w^{-1}A)^{\wedge}_{u_m} \cong ((w^{-1}MU_{\mathbb{Z}/n})_*)^{\wedge}_{u_m} \tag{21}$$

where w is a product of a set S of Euler classes of the form $u_{n/p_2}, \ldots, u_{n/p_s}$. When $|S| = s-1$, i.e. S contains all those classes, then in the ring

$$B = u_m^{-1}w^{-1}A,$$

all the Euler classes are inverted. The completion of $B[x, x_1, \ldots, x_{n-1}]$ at $\prod_{i=0}^{n-1} x_i$, by the Chinese remainder theorem, is

$$\prod_{\alpha \in \widehat{G}} B[[x_\alpha]]. \tag{22}$$

Now the relations (15) give an x-completed coproduct on $B[[x]]$, which we denote by $F(y,z)$ (i.e. we really have $y = x \otimes 1$, $z = 1 \otimes x$). The equivariant FGL axioms (2) and (3) imply that the coproduct on x_α must be

$$\prod_{\beta+\gamma=\alpha} \Delta(y_\beta, z_\gamma)$$

where $y_\alpha = x_\alpha \otimes 1$, $z_\alpha = 1 \otimes x_\alpha$. The relation (10) implies that this $\mathbb{Z}/n$-equivariant FGL definition is consistent, as well.

Also, by inspection of the relations,

$$B \cong \Phi^{\mathbb{Z}/n} MU_*.$$

(Since by [13], the coefficients b_{ij} of x^i in $x +_F u_j$ are free polynomial generators of $\Phi^{\mathbb{Z}/n}MU_*$, with $b_{0j} = u_j$ inverted, all we really need to show is that they are expressible in terms of the coefficients $\overline{a_{i'j'}}$ and u, which is obvious from the decomposition (22).)

Moreover, since the $\mathbb{Z}/n$-equivariant FGL's just defined on the corners of the diagram (18) are both induced from maps from the pullback A, they coincide when pushed forward to $u_m^{-1}((w^{-1}A^{\wedge}_{(u_m)})$. By reverse induction, we can then prove that

$$w^{-1}A \cong (w^{-1}MU_{\mathbb{Z}/n})_*$$

for any w as above, and in particular, it has a $\mathbb{Z}/n$-equivariant FGL on it. For $w = 1$, we have the ring A.

By general universal algebra, the compatible $\mathbb{Z}/n$-equivariant formal group laws on the three remaining corners of (18) define a $\mathbb{Z}/n$-equivariant FGL on A, which is induced, in the above sense, by $Id : A \to A$. It follows from the similar formal argument the $\mathbb{Z}/n$-equivariant FGL on A is universal.

□

It is also convenient to have a reformulation of this result in terms of an arbitrary complete flag (which is the formulation used in Theorem 1). This has two reasons. First of all, in the case of S^1, the analog of the regular flag is not complete. i.e. its union is not the complete (or any) universe. In fact, we will see below that the coefficients $\overline{a_{ij}}$ derived from the "regular flag" in the case of S^1 can be expressed merely as Laurent polynomials in the Euler classes, so they do not generate the coefficient ring of the universal S^1-equivariant FGL.

The other reason why having a flag-invariant description is desirable is of course aesthetic: there is no particular reason why the regular $\mathbb{Z}/n$-flag should be special. Of course, that flag is special in that there is a polynomial defining the Euler classes u_k in terms of $u = u_1$ and the coefficients $\overline{a_{ij}}$, which will not be true for an arbitrary complete flag, so we need to deal with this problem.

Let E be a complete $\mathbb{Z}/n$-flag. Define, for a $\mathbb{Z}/n$-equivariant formal group law F on a commutative ring A, elements $a_{ij}^E \in A$ by

$$\Delta(x) = \sum_{i,j\geq 0} a_{ij}^E x_{E_i} \otimes x_{E_j}. \tag{23}$$

Observe that if we write

$$y +_F z = \sum_{i,j\geq 0} a_{ij}^E y_{E_i} z_{E_j}, \tag{24}$$

then if we denote by u_k, $k = \{1, \dots, n-1\}$ the Euler classes of a $\mathbb{Z}/n$-equivariant FGL, expressions such as

$$x +_F u_i,\ u_i +_F u_j$$

make sense, since $(u_i)_{E_m} = 0$ for $m >> 0$. Using this, we can express the coefficients $a_{ij}^{E'}$ with respect to any other $\mathbb{Z}/n$-flag E' (complete or otherwise) in terms of the elements a_{ij}^E and the Euler classes u_k. This applies in particular to the "non-equivariant" FGL coefficients a_{ij}. Additionally, the relations

$$u_i +_F u_j = u_{i+j} \tag{25}$$

where the left hand side is interpreted as a polynomial in $u_k, a_{k\ell}^E$ for different k, ℓ (where $i, j, i+j \in \mathbb{Z}/n$, with the understanding that $u_0 = 0$). The case of Theorem 1 for the case of $G = \mathbb{Z}/n$ is then restated as follows:

Proposition 5. *The coefficient ring of the universal $\mathbb{Z}/n$-equivariant FGL can be expressed as the commutative ring with generators u_k, $1 \leq k \leq n-1$, a_{ij}^E, $i, j \in \mathbb{N}_0$ modulo the relations (15) (interpreted in terms of a_{ij}^E), and the relations (25).*

Proof. From the above, one can express these generators and derive these relations for any complete flag E' from the generators and relations for E. Since this is symmetrical, the presentations then give isomorphic rings for different complete flags. For the regular flag, however, the presentation is equivalent to the one we gave above (since $u_k +_F u$ gives an explicit formula for u_{k+1}, $k \geq 1$). This concludes the proof. $\square$

4. The S^1-case

The purpose of this section is to prove the following result, which is a restatement of Theorem 1 for $G = S^1$:

Theorem 6. *Let E be an S^1-complete flag. The coefficient ring of the universal S^1-equivariant FGL can be expressed as the commutative ring with generators u_k, $k \in \mathbb{Z} \smallsetminus \{0\}$, a^E_{ij}, $i, j \in \mathbb{N}_0$ modulo the relations (15) (interpreted in terms of a^E_{ij}), and the relations (25) for $i, j \in \mathbb{Z}$ (where we set $u_0 = 0$). Moreover, this ring supports the universal S^1-equivariant formal group law.*

Before proving this result, we begin with some general comments on S^1-equivariant spectra. Let, for $H \subseteq G$, $\mathcal{F}(H)$ denote the family of all subgroups of H. Then for $G = S^1$, we have

$$E\mathcal{F}[S^1] = \operatorname*{hocolim}_n E\mathcal{F}(\mathbb{Z}/n) \tag{26}$$

where the homotopy colimit is taken over the poset of natural numbers with the relation of divisibility. For any S^1-spectrum E, this gives a cofibration sequence

$$\operatorname*{hocolim}_n E\mathcal{F}(\mathbb{Z}/n) \wedge E \to E \to \widetilde{E\mathcal{F}[S^1]} \wedge E. \tag{27}$$

Taking S^1-fixed points, we get a cofibration sequence of spectra

$$\operatorname*{hocolim}_n (E^{\mathbb{Z}/n})_{hS^1} \to E^{S^1} \to \Phi^{S^1} E. \tag{28}$$

Here in the leftmost term, we are making the identification $S^1 \cong S^1/(\mathbb{Z}/n)$. But now we can complete (28) to a "Tate diagram" in whose bottom row, E is replaced by

$$F(E\mathcal{F}[S^1]_+, E) = \operatorname*{holim}_n F(E\mathcal{F}(\mathbb{Z}/n)_+, E).$$

The leftmost term (27) will remain the same, so we obtain a cofibration sequence

$$\operatorname*{hocolim}_n E\mathcal{F}(\mathbb{Z}/n) \wedge E \to \operatorname*{holim}_n F(E\mathcal{F}(\mathbb{Z}/n)_+, E) \to \widehat{E}_\infty \tag{29}$$

where

$$\widehat{E}_\infty = \operatorname*{hocolim}_m \operatorname*{holim}_n \widetilde{E\mathcal{F}(\mathbb{Z}/m)} \wedge F(E\mathcal{F}(\mathbb{Z}/n)_+, E) \tag{30}$$

and also a homotopy Cartesian diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \widetilde{E\mathcal{F}[S^1]} \wedge E \arrow[d] \\ \operatorname*{holim}_n F(E\mathcal{F}(\mathbb{Z}/n)_+, E) \arrow[r] & \widehat{E}_\infty. \end{tikzcd} \tag{31}$$

Taking S^1-fixed points, we have a homotopy Cartesian diagram of spectra

$$\begin{tikzcd} E^{S^1} \arrow[r] \arrow[d] & \Phi^{S^1} E \arrow[d] \\ \operatorname*{holim}_n (E^{\mathbb{Z}/n})^{hS^1} \arrow[r] & (\widehat{E}_\infty)^{S^1}. \end{tikzcd} \tag{32}$$

Further, for an S^1-complex oriented spectrum, $(\widehat{E}_\infty)^{S^1}$ is obtained from $\operatorname*{holim}_n (E^{\mathbb{Z}/n})^{hS^1}$ by inverting the Euler classes of all non-trivial irreducible S^1-representations.

Proof of Theorem 6: Let A be the ring with the presentation described in the statement of Theorem 6. By the comments before Proposition 5, for a commutative ring B, a homomorphism of rings

$$A \to B$$

determines all the data of an S^1-equivariant FGL on B (although we have not yet proved that this always gives, in fact, an S^1-equivariant FGL). However, if the data on A iself do determine an S^1-equivariant FGL, then as before, we know that it is universal.

Now to that end, we will study the algebraic counterpart of the diagram (32). Note that in A, we have $u_k \mid u_m$ when $k \mid m$. Consider a topology on A where a fundamental neighborhood of 0 is any system of ideals

$$\{(u_k) \mid k \in S\}$$

where S is a set of non-zero integers cofinal with respect to divisibility, i.e. such that for every non-zero integer k, there is an element of S which is divisible by k. Let $A^\wedge$ be the completion of A with respect to this topology.

On the algebraic side, we shall consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A[u_k^{-1} \mid k \in \mathbb{Z} \smallsetminus \{0\}] \arrow[d] \\ A^\wedge \arrow[r] & A^\wedge[u_k^{-1} \mid k \in \mathbb{Z} \smallsetminus \{0\}]. \end{tikzcd} \tag{33}$$

The plan is to map this diagram to the diagram obtained by taking coefficients of the terms of the diagram (32) for $E = MU$. This diagram from topology becomes

$$\begin{tikzcd} MU_*^{S^1} \arrow[r] \arrow[d] & MU_*[u_i, u_i^{-1}, b_{ij} \mid i \in \mathbb{Z} \smallsetminus \{0\}, j \in \mathbb{N}] \arrow[d] \\ (MU_*^{S^1})^\wedge \arrow[r] & (MU_*^{S^1})^\wedge[u_i^{-1} \mid i \in \mathbb{Z} \smallsetminus \{0\}] \end{tikzcd} \tag{34}$$

To identify the lower left term, recall that by the result of Sinha [12],

$$MU_*^{\mathbb{Z}/n} = MU_*^{S^1}/(u_n).$$

Additionally, by [12], all the elements u_n are non-torsion in $MU_*^{S^1}$, so by the Borel cohomology spectral sequence, the coefficients in the n term of the diagram in the lower left corner of (32) are

$$(MU_*^{S^1})^\wedge_{(u_n)}.$$

Our identification of the lower left corner coefficients then follows from the commutation of limits, and the fact that all the structure maps of the limit diagram on coefficients are onto.

Note that, thus, by Proposition 5, we have a canonical isomorphism of the lower left corner of (33) to the lower left corner of (34).

On the other hand, also recall that by [12], the top row of diagram (34) is localization by inverting all u_k, $k \in \mathbb{Z} \smallsetminus \{0\}$, and thus, since these elements are regular, is injective. We shall now construct an isomorphism from the top right hand corner of (33) to the top right hand corner of (34), and also prove that the top row of (33) is injective. To this end, we shall construct a map

$$A\{\{x_{E_i} \mid i \in \mathbb{N}_0\}\} \to \prod_{k \in \mathbb{Z}} A[[x_k]] \tag{35}$$

where on the left hand side, the double braces mean the product of copies of A over the given basis. Indeed, the map (35) is not difficult to construct simply by expanding the series on the left hand side in the given variable x_k.

Now order the basis elements of the factors on the right hand side of (35) as follows: Let

$$x_{E_i} = x_{k(i)} x_{E_{i-1}},\ i > 0.$$

Then let $m(i)$ be the exponent of $x_{k(i)}$ in x_{E_i}. Then let the i'th element of our basis B of the right hand side of (35) be

$$x_{k(i)}^{m(i)-1}.$$

Then by expanding x_ℓ in $x_{(k(i)}$, the base-change matrix between the basis (x_{E_i}) and the basis B is lower triangular with the diagonal coefficients momonials in the u_k's, $k \neq 0$. It follows that if we invert these u_k's, the matrix becomes invertible.

Additionally, however, the generators a_{ij}^E are just expansions of the formal group law $\Delta(x)$ in terms of the basis x_{E_i}. However, we know that on the right hand side of (35), the expansion of $\Delta(x)$ is just obtained in terms of the a_{ij}, b_{ij}, and u_k's. Therefore, we obtain an expression of each b_{ij} in terms of the $a_{i'j'}^E$'s, and, in return, the expression of a_{ij}^E times some monomial in the u_k's in terms of the $b_{i'j'}$, $a_{i'j'}$ and u_k's. This implies both that A cannot have any u_k-torsion, and that

$$A[u_k^{-1} \mid k \neq 0] \cong MU_*[u_i, u_i^{-1}, b_{ij} \mid i \neq 0, j \geq 0].$$

Thus, we have identified the upper right corner of the diagrams (33), (34), and by construction, the identifications of the lower left and upper right corners coincide on the lower right corner.

Now it is a result of Comezana [3] that the coefficients of $MU_*^{S^1}$ are concentrated in even degrees, and hence, the diagram (34) is a pullback. Thus, we obtain a homomorphism of rings

$$A \to MU_*^{S^1} \tag{36}$$

by universailty. On the other hand, the diagram (33) is always a pullback when the top row is injective. Therefore, (36) is an isomorphism.

But then this also implies that the data provided by the elements a_{ij}^E, u_k indeed define an S^1-equivariant formal group law on A, which, as already noted above, proves that this S^1-equivariant FGL is universal. □

It is interesting to add a comment on what happens to the regular flag V in the case of S^1 (of course in that case, by α, we mean the tautological irreducible representation). As already remarked, this flag is not complete, since it does not contain the representations α^n with $n > 0$, but also because it only contains one copy (instead of infinitely many copies) of each representation. Nevertheless, it is still possible to expand $\Delta(x)$ with respect to this flag, and therefore the elements $\overline{a_{ij}} \in MU_*^{S^1}$ exist.

Proposition 7. *The elements $\overline{a_{ij}} \in \Phi^{S^1} MU_*$ are Laurent polynomials with coefficients in $\mathbb{Z}$ in the Euler classes u_n, $n > 0$.*

Proof. Let $\mathcal{V}$ denote the union of the spaces V_i. Then the S^1-fixed points of the projective space $P(\mathcal{V})$ are identified with the infinite discrete set $\mathbb{N}_0$. Additionally,

(37)
$$(\widetilde{E\mathcal{F}[S^1]} \wedge MU)^* P(\mathcal{V}) = (\widetilde{E\mathcal{F}[S^1]} \wedge MU)^* (P(\mathcal{V}))^{S^1} = \prod_{n \in \mathbb{N}_0} \Phi^{S^1} MU^*.$$

Now analogously to the case of the complete flag, we have a lower triangular (topological) base change matrix from the basis x_{V_i} to the basis given by the generators of each factor on the right hand side of (37). This time, however, since we are only looking at the constant terms, the diagonal elements are only monomials in the u_n's, $n > 0$ (they do not even depend on the a_{ij}'s!). Since, on the other hand, the expansion of $\Delta(x)$ in terms of the topological basis on the right hand side of (37) are also the u_n's with $n > 0$, the statement follows. $\square$

We should comment that over $\mathbb{Z}/n$, the formulas given by Proposition 7 for $\overline{a_{ij}}$, $i, j < n$, are still valid. This is because when $\Delta(x)$ is expressed in terms of the basis B given by 1's in each factor on the right hand side of (37), we will get u_{i+j} in the (i,j)'th factor. Now the base-change matrix P from the basis $(x_{E_i})_{i \in \mathbb{N}_0}$ to the basis B is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & u_1 & 0 & 0 & \dots \\ 1 & u_2 & u_2 u_1 & 0 & \dots \\ 1 & u_3 & u_3 u_2 & u_3 u_2 u_1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

To get $\overline{a_{ij}}$, we need to base-change using the matrix $P^{-1} \otimes P^{-1}$. However, we will only need the first i rows of the first factor and the first j rows of the second factor. Therefore, we do not divide by any u_n for $n > \max(i,j)$. This can be proved by solving recursively for $\overline{a_{\ell m}}$ in the following generalization of formula (9):

$$u_k = u_\ell + u_m + \sum_{i=1}^{\ell} \sum_{j=1}^{m} \overline{a_{i,j}} (\prod_{s=0}^{i-1} u_{\ell - s})(\prod_{t=0}^{j-1} u_{m-t}).$$

When one of the indices is greater or equal to n, the formula will contain u_n denominators, which makes no sense, since $u_n = 0$. On the topological side, note that there is not direct functoriality between Φ^{S^1} and $\Phi^{\mathbb{Z}/n}$.

5. The case of a general abelian compact Lie group

We are now ready for the case of a general abelian compact Lie group G. Let E be a G-complete flag.

Theorem 8. *The ring MU_*^G is given by the generators a_{ij}^E and u_α, $\alpha \in \widehat{G}$ and the relations (15), (10),*

$$u_\alpha +_F u_\beta = u_{\alpha\otimes\beta}. \tag{38}$$

Moreover, this ring supports the universal G-equivariant formal group law.

Proof. It suffices to prove this result for a torus. This is because by [12], for a general compact Lie abelian group G embedded into a torus T, MU_*^G is known to be a quotient of MU_*^T by the relations on Euler classes given by the embedding $G \subset T$. For a torus, we prove the statement by induction on its rank. Let the statement be known for a torus T of a given rank m. Then we proceed to prove it by induction on n for $G = T \times \mathbb{Z}/n$. To this end, choose the flag E in such a way that for the representation α given by composing the projection $G \to \mathbb{Z}/n$ with the standard representation of $\mathbb{Z}/n$, E starts with

$$1, \alpha, \dots, \alpha^{n-1}.$$

That way, the stated generators and relations are equivalent to a set of generators and relations containing the Euler classes w_γ of representations γ of T, and the Euler class u of α. The relations (38) are included only for the representatinos w_γ, and there is an additional relation (10) (since $u_2, \cdots u_n$ can again be expressed using (9)). Now an analog of Lemma 4, and the conclusion of the proof of Theorem 3, can be proved the same way, replacing MU_* by MU_*^T. (Note that by [12], $MU_*^T \subset \Phi^T MU_*$ is an integral domain.)

□

References

[1] M.Cole, J.P.C.Greenlees, I.Kriz: Equivariant formal group laws, *Proc. London Math. Soc.* 81 (2000), 355-386

[2] M.Cole, J.P.S.Greenlees, I.Kriz: The universality of equivariant complex bordism, *Math Z.* 239 (2002) 455-475

[3] J. P. May: Equivariant homotopy and cohomology theory, volume 91 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. With contributions by M. Cole, G. Comeza na, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Trianta
llou, and S. Waner., Chapters XXVII, XXVIII

[4] J. P. C. Greenlees and J. P. May. *Generalized Tate cohomology.* Memoirs of the AMS, 543 (1995).

[5] B.Hanke, M.Wiemeler: An equivariant Quillen theorem, *Adv. Math.* 340 (2018), 48-75

[6] M.Hausmann: Global group laws and equivariant bordism rings, arXiv:1912.07583

[7] P.Hu: The coefficients of equivariant complex cobordism for primary cyclic groups, to appear

[8] I.Kriz: The $\mathbb{Z}/p$-equivariant stable complex cobordism ring, Homotopy invariant algebraic structures (Baltimore, MD, 1998), 217–223, *Contemp. Math.*, 239, Amer. Math. Soc., Providence, RI, 1999

[9] M. Lazard: Sur les groupes de Lie Formels un Parametre, *Bull. Soc. France* 83 (1955)

[10] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure: *Equivariant stable homotopy theory*, Vol 1213 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1986

[11] N.P.Strickland: Complex cobordism of involutions, *Geom. Top.* 5 (2001) 335-345

[12] D.P. Sinha: Computations of complex equivariant bordism rings. *Amer. J. Math.* 123 (2001), no. 4, 577-605

[13] T. tom Dieck: Bordism of G-manifolds and integrality theorems. *Topology* 9, (1970), no. 4, 345-358

[14] T. tom Dieck: Kobordismentheorie und Transformationsgruppen, Aaarhus University Preprint Series (1968/69) No. 30