1 Homeomorphisms

Definition 1.1. (Homeomorphisms of topological spaces). Let $(X, T_X)$ and $(Y, T_Y)$ be topological spaces. A map $f : X \to Y$ is a homeomorphism if

- $f$ is continuous,
- $f$ has an inverse $f^{-1} : Y \to X$, and
- $f^{-1}$ is continuous.

The topological space $(X, T_X)$ is said to be homeomorphic to the topological space $(Y, T_Y)$ if there exists a homeomorphism $f : X \to Y$.

Two topological spaces are considered “the same” topological space if and only if they are homeomorphic.

In-class Exercises

1. (a) Let $(X, T_X)$ be a topological space. Show that $X$ is homeomorphic to itself.
   
   (b) Let $(X, T_X)$ and $(Y, T_Y)$ be topological spaces, and $f : X \to Y$ a homeomorphism. Explain why $f^{-1} : Y \to X$ is also a homeomorphism. Conclude that $X$ is homeomorphic to $Y$ if and only if $Y$ is homeomorphic to $X$. (We simply call the spaces “homeomorphic topological spaces”).
   
   (c) Let $(X, T_X), (Y, T_Y),$ and $(Z, T_Z)$ be topological spaces. Show that, if $X$ is homeomorphic to $Y$, and $Y$ is homeomorphic to $Z$, then $X$ is homeomorphic to $Z$.

   This exercise shows that homeomorphism defines an equivalence relation on topological spaces.

2. (a) Give an example of topological spaces $(X, T_X)$ and $(Y, T_Y)$ and a map $f : X \to Y$ such that $f$ is both continuous and invertible, but such that $f^{-1}$ is not continuous.
   
   (b) Let $(X, T_X)$ and $(Y, T_Y)$ be topological spaces, and $f : X \to Y$ a map. Show that $f$ is a homeomorphism if and only if it is a continuous, invertible, open map.

3. Determine which of the following properties are preserved by homeomorphism. In other words, suppose $(X, T_X)$ and $(Y, T_Y)$ are homeomorphic topological spaces. For each of the following properties $P$, prove or give a counterexample to the statement “$X$ has property $P$ if and only if $Y$ has property $P$.”

   (For some properties, you will need to assume that $X$ and $Y$ are metric spaces.)

   (i) discrete topology   (vii) complete
   (ii) indiscrete topology   (viii) sequentially compact
   (iii) Hausdorff   (ix) compact
   (iv) regular
   (v) number of connected components   (x) bounded
   (vi) path-connected   (xi) metrizable

   Properties that are preserved by homeomorphisms are called homeomorphism invariants, topological invariants, or topological properties of a topological space.
4. Use the results of Problem 3 to explain why the following pairs of spaces are not homeomorphic.
   (a) $(0, 1)$ and $[0, 1]$ (with the Euclidean metric)
   (b) $\mathbb{R}$ with the Euclidean metric and $\mathbb{R}$ with the cofinite topology
   (c) $(0, 2)$ and $(0, 1] \cup (2, 3)$ (with the Euclidean metric)

5. (Bonus) Let $(X, T_X)$ and $(Y, T_Y)$ be topological spaces, and $F : X \to Y$ a continuous function. Let $G$ denote the graph of $F$ (as a subspace of $X \times Y$). Prove that $G$ is homeomorphic to $X$. 