1 Interior, closure, and boundary

Recall the definitions of interior and closure from Homework #7.

**Definition 1.1. (Interior of a set in a topological space).** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Define the interior of $A$ to be the set

$$\text{Int}(A) = \{ a \in A \mid \text{there is some neighbourhood } U \text{ of } a \text{ such that } U \subseteq A \}.$$ 

You proved the following:

**Proposition 1.2.** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

- $\text{Int}(A)$ is an open subset of $X$ contained in $A$.
- $\text{Int}(A)$ is the largest open subset of $A$, in the following sense: If $U \subseteq A$ is open, then $U \subseteq \text{Int}(A)$.

**Definition 1.3. (Closure of a set in a topological space).** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Define the closure of $A$ to be the set

$$\overline{A} = \{ x \in X \mid \text{any neighbourhood } U \text{ of } x \text{ contains a point of } A \}.$$ 

You proved the following:

**Proposition 1.4.** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

- $\overline{A}$ is a closed subset containing $A$.
- $\overline{A}$ is the smallest closed subset containing $A$, in the following sense: If $C$ is a closed subset with $A \subseteq C$, then $\overline{A} \subseteq C$.

We can similarly define the boundary of a set $A$, just as we did with metric spaces.

**Definition 1.5. (Boundary of a set $A$).** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then the boundary of $A$, denoted $\partial A$, is the set $\overline{A} \setminus \text{Int}(A)$.

**Example 1.6.** Find the interior, closure, and boundary of the following subsets $A$ of the topological spaces $(X, \mathcal{T})$.

(a) $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, \{c\}, \{b, c\}, \{a, b, c\}\}$, $A = \{a, c\}$.

(b) $X = \mathbb{R}$ with the cofinite topology, $A = (0, 1)$. 


In-class Exercises

1. Let $X$ be a topological space, and $A \subseteq X$. Prove the following.
   (a) $A$ is open if and only if $A = \text{Int}(A)$.
   (b) $\text{Int}(\text{Int}(A)) = \text{Int}(A)$.
   (c) $\text{Int}(A) = \bigcup_{U \subseteq A, U \text{ open}} U$.

2. Let $X$ be a topological space, and $A \subseteq X$. Prove the following.
   (a) $A$ is closed if and only if $A = \overline{A}$.
   (b) $\overline{A} = \overline{A}$.
   (c) $\overline{A} = \bigcap_{C \subseteq X, C \text{ closed}, A \subseteq C} C$.

3. Let $X$ be a topological space, and $A \subseteq X$.
   (a) Prove that $\partial A = \overline{A} \cap (X \setminus A)$.
   (b) Use this result to conclude that (i) $\partial A$ is closed, and (ii) $\partial A = \partial(X \setminus A)$.
   (c) Prove the following.
   \begin{align*}
   \text{Theorem (An equivalent definition of $\partial A$).} \quad &\text{Let } X \text{ be a topological space, and let } A \subseteq X. \text{ Then} \\
   \partial A = \left\{ x \in X \,\big|\, \text{every open neighbourhood } U \text{ of } x \text{ contains at least one point of } A, \right. \\
   &\left. \quad \text{and at least one point of } X \setminus A \right\} \\
   \end{align*}
   (d) Prove that every point of $X$ falls into one of the following three categories of points, and that the three categories are mutually exclusive:
   (i) interior points of $A$;
   (ii) interior points of $X \setminus A$;
   (iii) points in the (common) boundary of $A$ and $X \setminus A$.

4. (Optional). Let $A$ be a subset of a topological space $X$. Prove the following.
   (a) $X \setminus \overline{A} = \text{Int}(X \setminus A)$.
   (b) $X \setminus \text{Int}(A) = \overline{X \setminus A}$.

5. (Optional). Suppose $(X, d)$ is a metric space, and $A \subseteq X$. You proved on Quiz #4 that, if $x \in \overline{A}$, then there is some sequence of points $(a_n)_{n \in \mathbb{N}}$ in $A$ that converge to $x$. In this problem, we will see that this property does not hold for general topological spaces.
   (a) Recall that the co-countable topology on $\mathbb{R}$ is
   \[ \mathcal{T}_{cc} = \{ \emptyset \} \cup \left\{ U \subseteq \mathbb{R} \,|\, \mathbb{R} \setminus U \text{ is countable} \right\}. \]
   Let $A \subseteq \mathbb{R}$. What is $\overline{A}$ if $A$ is (i) countable, or (ii) uncountable?
   (b) Let $A = (0, 1)$, so $\overline{A} = \mathbb{R}$. Show that, for any $x \in \overline{A} \setminus A$, there is no sequence of points in $A$ that converge to $x$.
   (c) \text{Definition (First countable spaces).} A topological space $(X, \mathcal{T})$ is called \textit{first countable} if each point $x \in X$ has a \textit{countable neighbourhood basis}. This means, for each $x \in X$, there is a countable collection $\{ N_i \}_{i \in \mathbb{N}}$ of neighbourhoods of $x$ with the property that, if $N$ is any neighbourhood of $x$, then there is some $i$ such that $N_i \subseteq N$.
   Let $X$ be a first countable space, and let $A \subseteq X$. Show that, given any $x \in \overline{A}$, there is some sequence of points in $A$ that converges to $x$. 