1 Products of compact spaces

In this handout, we will prove the following theorem.

**Theorem 1.1. (Products of compact spaces).** Let \((X, T_X)\) and \((Y, T_Y)\) be nonempty topological spaces. Then \(X \times Y\) is compact with respect to the product topology \(T_{X \times Y}\) if and only if both \(X\) and \(Y\) are compact.

**In-class Exercises**

1. Let \((X, T_X)\) and \((Y, T_Y)\) be two nonempty topological spaces. Suppose that their Cartesian product \(X \times Y\) is compact with respect to the product topology \(T_{X \times Y}\). Prove that \(X\) and \(Y\) are compact.

2. Let \((X, T_X)\) and \((Y, T_Y)\) be nonempty compact topological spaces. Let \(U\) be any open cover of \(X \times Y\) (with the product topology).
   For this exercise, we will call a subset \(A \subseteq X\) *good* if \(A \times Y\) is covered by a finite subcollection of open sets in \(U\). Our goal is to show that \(X\) is good.
   (a) Suppose that \(A_1, \ldots, A_r\) is a finite collection of good subsets of \(X\). Show that their union is good.
   (b) Fix \(x \in X\). For each \(y \in Y\), explain why it is possible to find open sets \(U_y \in X\) and \(V_y \in Y\) so that \((x, y) \in U_y \times V_y\) and \(U_y \times V_y\) is contained in some open set in \(U\).
   (c) Explain why there is a finite list of points \(y_1, \ldots, y_n \in Y\) so that the sets \(\{V_{y_1}, \ldots, V_{y_n}\}\) cover \(Y\).
   (d) Define \(U_x = U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}\).
      Show that \(U_x\) is a good set, and is an open subset of \(X\) containing \(x\). This shows that every element \(x \in X\) is contained in a good open set \(U_x\).
   (e) Consider the collection of open subsets \(\{U_x \mid x \in X\}\) of \(X\). Use the fact that \(X\) is compact to conclude that \(X\) is good.

3. (Optional).
   **Definition (Lindelöf).** A topological space \(X\) is called *Lindelöf* if every open cover of \(X\) has a countable subcover.

   Suppose that \(X\) is a Lindelöf space and \(Y\) is a compact space. Prove that the product \(X \times Y\), with the product topology, is Lindelöf.

4. (Optional). Recall that a map of topological spaces is called *closed* if the image of every closed set in the domain is a closed subset of the codomain.

   Let \(X\) and \(Y\) be topological spaces, and endow their product \(X \times Y\) with the product topology. We saw on Worksheet #7 Problem 4 that the projection map \(\pi_X : X \times Y \to X\) need not be closed in general. Prove that, if \(Y\) is compact, then \(\pi_X\) is a closed map.