1 Continuous functions on metric spaces

Definition 1.1. (Continuous functions \( f : \mathbb{R} \to \mathbb{R} \).) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. Then \( f \) is continuous at a point \( x \in \mathbb{R} \) if . . .

The function \( f \) is called continuous if it is continuous at every point \( x \in \mathbb{R} \).

Rephrased:

How can we generalize this definition to general metric spaces?

Definition 1.2. (Continuous functions on metric spaces.) Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. Let \( f : X \to Y \) be a function. Then \( f \) is continuous at a point \( x \in X \) if . . .

The function \( f \) is called continuous if it is continuous at every point \( x \in X \).

Rephrased:
In-class Exercises

1. In this question, we will prove the following result:

**Theorem (Continuous functions.)** Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces, and let $f : X \to Y$ be a function. Then $f$ is continuous if and only if, given any open set $U \subseteq Y$, its preimage $f^{-1}(U) \subseteq X$ is open.

(a) Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces, and let $f : X \to Y$ be a continuous function. Suppose that $U \subseteq Y$ is an open set. Prove that $f^{-1}(U)$ is open.

(b) Suppose that $f$ is a function with the property that, for every open set $U \subseteq Y$, the preimage $f^{-1}(U)$ is an open set in $X$. Show that $f$ is continuous.

2. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions between metric spaces. Show that the composite

$$g \circ f : X \to Z$$

is continuous. **Hint:** With our new criterion for continuity, this argument can be quite quick!

3. *(Optional)* Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces. A function $f : X \to Y$ is called an isometric embedding if it is “distance-preserving” in the sense that

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

(a) Give an intuitive description, with pictures, of what it means for a map to be an isometric embedding.

(b) Show that an isometric embedding is continuous.

(c) Show that an isometric embedding is always injective.

(d) Show that map

$$f : \mathbb{R} \longrightarrow \mathbb{R}^2$$

$$x \longmapsto (x, 0)$$

is an isometric embedding of $\mathbb{R}$ into $\mathbb{R}^2$ (each with the Euclidean metric).

(e) Consider the linear function $f : \mathbb{R} \to \mathbb{R}^2$ given by the map $f(x) = (x, mx + b)$ for $m, b \in \mathbb{R}$, so $f$ maps the real line to the graph of the function $mx + b$. For which values of $m$ and $b$ is this an isometric embedding of Euclidean spaces?

(f) For functions $f(x) = (x, mx+b)$ that are not isometric embeddings, can you find a different parameterization of this line that is an isometric embedding? In other words, can you find an isometric embedding $g : \mathbb{R} \to \mathbb{R}^2$ whose image is the set $\{(x, mx + b) \mid x \in \mathbb{R}\}$?

(g) Let $X = \{a, b, c\}$ be a 3-point set. Find examples of metrics on $X$ so that the resulting metric space can and cannot be isometrically embedded in Euclidean space $\mathbb{R}^2$. Can you find necessary and sufficient conditions on the metric on $X$ to guarantee the existence of an isometric embedding of $X$ into $\mathbb{R}^2$?

(h) Show that the image of any isometric embedding from $\mathbb{R}$ into $\mathbb{R}^2$ must be a straight line.