

## Representation Stability and FI-Modules

Exposition on work by Church – Ellenberg – Farb

Jenny Wilson

University of Chicago  
wilsonj@math.uchicago.edu

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This talk was given at the 2012 Topology Student Workshop, held at the Georgia Institute of Technology from June 11–15, 2012.

It is primarily an expository talk on recent work done by my advisor, Benson Farb, with Tom Church and Jordan Ellenberg.

Most of the talk will be devoted to defining the terms in the title.

## Background: Classical Homological Stability

$\{Y_n\}_n$  is a sequence of groups or topological spaces,  
with inclusions

$$\phi_n : Y_n \rightarrow Y_{n+1}$$

### Definition (Homological Stability)

The sequence  $\{Y_n\}$  is *homologically stable* (over a ring  $R$ ) if for each  $k \geq 1$ , the map

$$(\phi_n)_* : H_k(Y_n; R) \rightarrow H_k(Y_{n+1}; R)$$

is an isomorphism for  $n \gg k$ .

The context for the project “Representation Stability and FI-Modules” is the study of homological stability.

Suppose we have a sequence of groups or spaces, with inclusions. Fix a nonnegative integer  $k$ . Eventually, for  $n$  large enough relative to  $k$ , we want the induced maps on  $k^{\text{th}}$  homology to be isomorphisms. If this property holds for every  $k$ , we say that the sequence is homologically stable.

There are similar definitions for cohomology groups, homotopy groups, ...

## Examples of Homologically Stable Sequences

- (Nakaoka 1961)  
**Symmetric groups  $S_n$**
- (Arnold 1968, Cohen 1972)  
**Braid groups  $B_n$**
- (McDuff 1975, Segal 1979)  
**Configuration spaces of open manifolds**
- (Charney 1979, Maazen 1979, van der Kallen 1980)  
**Linear groups, arithmetic groups (such as  $SL_n(\mathbb{Z})$ )**
- (Harer 1985)  
**Mapping class groups of surfaces with boundary**
- (Hatcher 1995)  
**Automorphisms of free groups  $\text{Aut}(F_n)$**
- (Hatcher–Vogtmann 2004)  
**Outer automorphisms of free groups  $\text{Out}(F_n)$**

Here is a sampling of major results from the past 50 years.

This list, and the references, are by no means comprehensive – but they should give an idea of how important, and pervasive, this phenomenon is in topology and geometry.

## Generalizing Homological Stability

### What can we say when $H_k(Y_n; R)$ does not stabilize?

More generally, let  $\{V_n\}_n$  be a sequence of  $R$ -modules. Suppose  $V_n$  has an action by a group  $G_n$ .

Our objective: A notion of stability for  $\{V_n\}_n$  that takes into account the  $G_n$ -symmetries.

In this talk:

- $G_n = S_n$ , the symmetric group
- $R = \mathbb{Q}$ , and  $V_n$  are finite dimensional vector spaces

Two years ago, Church and Farb released a preprint about what they called "representation stability". Two months ago, Church, Ellenberg, and Farb released a second preprint extending this work, using the concept of an "FI-module".

A major goal of these papers is to develop a framework to study sequences that *do not* stabilize in the sense of homological stability.

Consider the following setup: Suppose we have a sequence of  $R$ -modules  $\{V_n\}_n$  (for example,  $V_n = H_k(Y_n; R)$ ), and suppose  $V_n$  has an action of some group  $G_n$ .

We want to use the extra structure encoded in these group actions to study the long-term behaviour of the sequence. The objective is to develop a concept of stability in terms of the  $G_n$ -symmetries.

For this talk, for simplicity, we will restrict our attention to the case when the groups  $G_n$  are the symmetric groups  $S_n$ , and  $\{V_n\}_n$  is a sequence of finite-dimensional rational vector spaces.

## An Example: The Permutation Representation

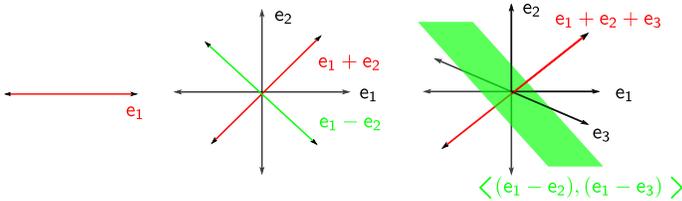
### Example (The Permutation Representation)

Consider the permutation representation

$$V_n = \mathbb{Q}^n = \langle e_1, \dots, e_n \rangle.$$

For each  $n$ ,  $V_n$  decomposes into two irreducibles:

$$\mathbb{Q}^n = \left\langle a(e_1 + e_2 + \dots + e_n) \right\rangle \oplus \left\langle a_1 e_1 + \dots + a_n e_n \mid \sum a_i = 0 \right\rangle$$



We will begin with a toy example of a sequence that exhibits the sort of stability we want.

Consider the permutation representations, the usual action of the symmetric group  $S_n$  on  $\mathbb{Q}^n$  by permutation matrices.

Each representation decomposes into two invariant subspaces. The 1-dimensional diagonal subspace (shown in red) is fixed by the action. For each  $n \geq 2$ , the diagonal has an  $(n - 1)$ -dimensional invariant complement, the subspace of coefficient-sum-zero (shown in green). This complement is also irreducible – it is isomorphic to the "standard representation".

## An Example: The Permutation Representation

### Some properties of the permutation representation

- The decomposition into irreducibles 'looks the same' for every  $n$ .

$$\mathbb{Q}^n = \left\langle a(e_1 + e_2 + \dots + e_n) \right\rangle \oplus \left\langle a_1 e_1 + \dots + a_n e_n \mid \sum a_i = 0 \right\rangle$$

- The dimension of  $V_n$  grows polynomially in  $n$

$$\dim(V_n) = n$$

- The characters  $\chi_n$  of  $V_n$  have a 'nice' global description

$$\chi_n(\sigma) = \#1\text{-cycles of } \sigma \quad \text{for all } \sigma \in S_n, \text{ for all } n.$$

Let's highlight some properties of these permutation representations.

Notice that the sequence is *not* stable in the sense that the spaces are eventually isomorphic – in fact, the dimensions grow without bound. However, there are several things we can say about the sequence.

Firstly, heuristically, the description of the irreducible subrepresentations is similar for each  $n$ . In a moment, I'll give a precise definition of what it means for decompositions into irreducibles to 'look the same' as  $n$  grows.

Secondly, the dimension of  $V_n$  is polynomial in  $n$ .

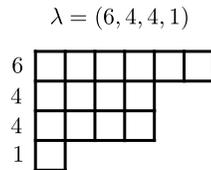
Thirdly, the characters of these representations can be described uniformly in  $n$ . Given a permutation matrix, its trace is simply the number of fixed points of the permutation. In other words, for any permutation for any value of  $n$ , the character of that permutation is equal to the number of 1-cycles in its cycle type. Again, I will soon give a precise statement of what I mean by a 'uniform description' for a sequence of  $S_n$  characters.

## Some Representation Theory

### Some facts about $S_n$ -representations over $\mathbb{Q}$

- Every  $S_n$ -representation decomposes uniquely as a sum of irreducibles.

- Irreducibles are indexed by partitions  $\lambda$  of  $n$ , depicted by *Young diagrams*.



In order to state what it means for the decomposition into irreducible representations to 'look the same' for different values of  $n$ , let's recall some representation theory of the symmetric group.

Recall that a representation of  $S_n$  is *irreducible* if it has no nontrivial  $S_n$ -stable subspaces. Rational representations decompose completely and uniquely into a sum of irreducibles.

There is a canonical way of constructing the irreducible representations of  $S_n$ , indexed by partitions of  $n$ . A partition of  $n$  is a list of positive integers that sum to  $n$  – we denote these by Young diagrams, with each integer encoded by the length of a row. An example is given for when  $n = 15$ .

### Obstacle

How can we compare irreducibles for different values of  $n$ ?

### Solution

Two irreducibles are "the same" if only the top rows of their Young diagrams differ.

### Example (The Permutation Representation $V_n = \mathbb{Q}^n$ )

$$\mathbb{Q}^1 = V_{\square}$$

$$\mathbb{Q}^2 = V_{\square\square} \oplus V_{\square}$$

$$\mathbb{Q}^3 = V_{\square\square\square} \oplus V_{\square\square}$$

$$\mathbb{Q}^4 = V_{\square\square\square\square} \oplus V_{\square\square\square}$$

$$\mathbb{Q}^5 = V_{\square\square\square\square\square} \oplus V_{\square\square\square\square}$$

Given this classification of irreducibles  $S_n$ -representations, we're immediately faced with an obstacle: the irreducibles are indexed by partitions of  $n$ , which, of course, depend on  $n$ . It's not clear how to define identifications between irreducibles for different values of  $n$ .

The solution: we declare that two irreducibles (for different values of  $n$ ) are "the same" if only the top row of their Young diagrams differ.

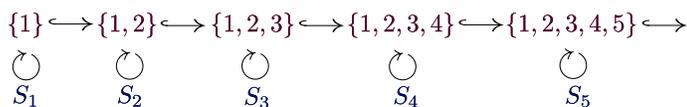
To illustrate what this means, observe the decomposition of the permutation representations into irreducibles. Once  $n \geq 2$ , we get the decomposition at stage  $(n + 1)$  by 'adding a box' to the top row of each of Young diagram at stage  $n$ .

This procedure of 'adding a box to the top row' of each irreducible is exactly the pattern that defines *representation stability* for a sequence of representations of the symmetric groups.

## The Definition of an FI-module

### Definition (Church–Ellenberg–Farb) (The Category FI)

Denote by FI the category of Finite sets with Injective maps



We are now ready for our main definition, of an FI-module.

To begin, we define the category FI – here, FI stands for "finite sets" and "injective maps".

We can realize this category as follows: The objects are indexed by the natural numbers – the number  $n$  corresponds to the set of numbers  $\{1, 2, \dots, n\}$ . The morphisms are all injective maps between these sets.

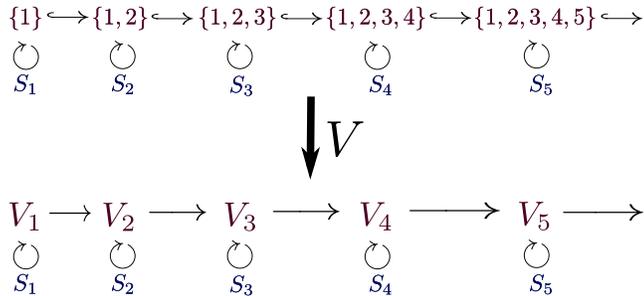
Notice, in particular, that the endomorphisms of the object  $n$  are the symmetric group  $S_n$ .

## The Definition of an FI-module

### Definition (Church–Ellenberg–Farb) (FI-Modules)

A (rational) FI-module is a functor

$$V : \text{FI} \rightarrow \mathbb{Q}\text{-Vect}$$



We define a rational FI-module to be a functor from the category FI to the category of rational vector spaces.

What is the data of an FI-module? For each  $n$ , we have a vector space  $V_n$ , with an action of the symmetric group  $S_n$ . Additionally, we have a host of linear maps between these vector spaces, which are compatible with these  $S_n$ -actions.

As a first example, it's an exercise to verify that we can give the sequence of permutation representations the structure of an FI-module, by appropriately defining linear maps between the vector spaces.

## Finite Generation of FI-Modules

### Definition (Generation)

If  $V$  is an FI-module, and  $S \subseteq \coprod_n V_n$ , then the FI-module generated by  $S$  is the smallest sub-FI-module containing the elements of  $S$ .

### Definition (Finite Generation)

An FI-module is *finitely generated* if it has a finite generating set.

### Example (The Permutation Representation $V_n = \mathbb{Q}^n$ )

The permutation representation  $V_n = \mathbb{Q}^n = \langle e_1, \dots, e_n \rangle$  is generated by  $e_1 \in V_1$ .

We can define 'generation' of FI-modules in the usual way.

Given an FI-module  $V$ , and a set  $S$  of vectors from the various vector spaces  $V_n$ , take all linear combinations of all images of these vectors under the induced linear maps – these spaces themselves comprise an FI-module, which we say is the FI-module *generated* by  $S$ .

Equivalently, the FI-module generated by  $S$  is the smallest sub-FI-module that contains all the vectors in  $S$ .

Having defined generation, we have a notion of what it means for an FI-module to be finitely generated.

It is an exercise to verify that our sequence of permutation representations, as an FI-module, is generated by the single basis element  $e_1$  in  $V_1$ .

## Consequences of Finite Generation

### Theorem (Church–Ellenberg–Farb)

Let  $V$  be a finitely-generated FI-module. Then for  $n \gg 1$

- The decomposition into irreducible  $S_n$ -representations stabilizes.
- $\dim(V_n)$  is polynomial in  $n$
- The characters  $\chi_n$  of  $V_n$  are given by a (unique) polynomial in the variables  $X_r$

$$X_r(\sigma) = \#r\text{-cycles of } \sigma \quad \text{for all } \sigma \in S_n, \text{ for all } n.$$

Any sub-FI-module of  $V$  also has these properties.

We call the sequence  $\{V_n\}_n$  *uniformly representation stable*.

Now, the main theorem.

If we have a finitely generated FI-module  $V$ , then the underlying sequence of  $S_n$ -representations satisfies the following properties, for large  $n$ :

Firstly, the decomposition into irreducible representations stabilizes – in the sense that we can recover  $V_{n+1}$  from  $V_n$  by 'adding a box to the top row' to the Young diagram for each irreducible.

Secondly, the dimension of  $V_n$  is polynomial in  $n$ .

Thirdly, the characters have a 'global' description: they are polynomials in variables  $X_r$ , where  $X_r$  is a function that takes a permutation and reads off the number of  $r$ -cycles in its cycle type.

The theorem actually says more: in a given example, we can put enough constraints on these character polynomials that the problem of determining the characters is reduced to a finite computation. It is enough to compute the value of the characters for certain small  $n$ .

Finally, these FI-modules have the 'Noetherian property' that every sub-FI-module of a finitely generated FI-module is itself finitely generated, and so also satisfies all of these properties.

## Some Representation Stable Cohomology Sequences

(Church–Farb)	$\{H^k(P_n; \mathbb{Q})\}_n$	<b>The pure braid group</b>
(Jimenez-Rolland)	$\{H^k(\text{PMod}(\Sigma_{g,r}^n); \mathbb{Q})\}_n$	<b>The pure MCG of an <math>n</math>-puncture surface <math>\Sigma_{g,r}^n</math></b>
(Church)	$\{H^k(\text{PConf}_n(M); \mathbb{Q})\}_n$	<b>Ordered configuration space of a manifold <math>M</math></b>
(Putman)	$\{H^k(\text{PMod}^n(M); \mathbb{F})\}_n$	<b>The pure MCG of an <math>n</math>-puncture manifold</b>
(Putman)	Eg, $\{H^k(\text{SL}_n(\mathbb{Z}, \ell); \mathbb{F})\}_n$	<b>Certain congruence subgroups</b>
(Wilson)	$\{H^k(P\Sigma_n; \mathbb{Q})\}_n$	<b>The pure symmetric automorphism group <math>P\Sigma_n</math> of the free group</b>

Here are some results that have been proven since Church–Farb’s “representation stability” paper appeared 2 years ago.

The FI–module ‘machinery’ developed by Church–Ellenberg–Farb earlier this year has hugely simplified many of these proofs. These results were originally proved through a detailed analysis of the combinatorics of the decomposition into  $S_n$ –irreducibles – a much more difficult task than verifying an FI–module structure and proving finite generation.

The ‘Noetherian property’ has also proven a powerful tool. In particular, to study cohomology of the mapping class group of a punctured manifold, or of the configuration space of a manifold, we realize these cohomology groups as the limit of a spectral sequence. The ‘Noetherian property’ implies that it suffices to prove finite generation for the  $E_2$  page, which enormously simplifies these proofs.

In summary: Simple (and often easily-verified) symmetries of these sequences of  $S_n$ –representations imposes very strong constraints on their structure and growth. The failure of these sequences to be (co)homologically stable in the classical sense can be seen as a consequence of these symmetries.

## Open Question

### Problem

Compute the characters, and the stable decompositions into irreducibles, in the above examples.

In each of these examples, it has been proven that for each  $k$ , the cohomology groups are uniformly representation stable – but we know very little about what the stable characters, and decompositions into irreducibles, actually are.

These questions are particularly compelling, since the constraints on the degree of the character polynomials make these computations much more tractable.

## My Research

### My current project

To develop a unified “FI–module theory” for the three families of classical Weyl groups.

FI–modules are a tool for studying sequences of representations of the symmetric groups. I am currently working on a project to extend the theory to sequences of representations of the other classical families of Weyl groups – the hyperoctahedral groups in types B/C, and the even-signed permutation groups in type D.

## References

### Further Reading

T Church, B Farb.

*Representation theory and homological stability*, preprint, 2010.

T Church, J Ellenberg, B Farb.

*FI-modules: A new approach to stability for  $S_n$ -representations*, preprint, 2012.

In this talk, I described representation stability and FI-modules as tools to generalize the theory of homological stability to sequences with group actions. This is, in fact, only one corner of the theory that has been developed by Church–Ellenberg–Farb and others.

The theory also has applications to classical representation theory, combinatorics, algebraic geometry, and number theory, among other fields. There are a wealth of open problems and possible new directions.

Full details and additional applications are given in these preprints by Church–Farb and Church–Ellenberg–Farb.

Both papers are available on the ArXiv.

The End

### Acknowledgements

Many thanks to Benson Farb, Tom Church, Jordan Ellenberg, and Rita Jimenez-Rolland for their help and guidance.

### In summary

- The theory of representation stability and FI-modules gives a language for understanding the patterns of growth of sequences of group representations.
- It allows us to deduce strong constraints on the structure of such a sequence using only elementary symmetries.

Comments, questions, and suggestions are welcome.

Contact me at [wilsonj@math.uchicago.edu](mailto:wilsonj@math.uchicago.edu).