Representation Stability and FI–Modules

Exposition on work by Church – Ellenberg – Farb

Jenny Wilson

University of Chicago

wilsonj@math.uchicago.edu

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Background: Classical Homological Stability

$\{Y_n\}_n$ is a sequence of groups or topological spaces,

with inclusions

$$\phi_n \ : \ Y_n \to Y_{n+1}$$

Definition (Homological Stability)

The sequence $\{Y_n\}$ is *homologically stable* (over a ring R) if for each $k \geq 1$, the map

$$(\phi_n)_* \ : \ H_k(Y_n; R) \to H_k(Y_{n+1}; R)$$

is an isomorphism for $n >> k$.

Examples of Homologically Stable Sequences

- (Nakaoka 1961)
  Symmetric groups S_n
- (Arnold 1968, Cohen 1972)
  Braid groups B_n
- (McDuff 1975, Segal 1979)
  Configuration spaces of open manifolds
- (Charney 1979, Maazen 1979, van der Kallen 1980)
  Linear groups, arithmetic groups (such as $SL_n(\mathbb{Z})$**)**
- (Harer 1985)
  Mapping class groups of surfaces with boundary
- (Hatcher 1995)
  Automorphisms of free groups Aut(F_n)
- (Hatcher–Vogtmann 2004)
  Outer automorphisms of free groups Out(F_n)

Generalizing Homological Stability

What can we say when $H_k(Y_n; R)$ does not stabilize?

More generally, let $\{V_n\}_n$ be a sequence of R-modules.
Suppose V_n has an action by a group G_n.

Our objective: A notion of stability for $\{V_n\}_n$ that takes into account the G_n–symmetries.

In this talk:

- $G_n = S_n$, the symmetric group
- $R = \mathbb{Q}$, and V_n are finite dimesional vector spaces

An Example: The Permutation Representation

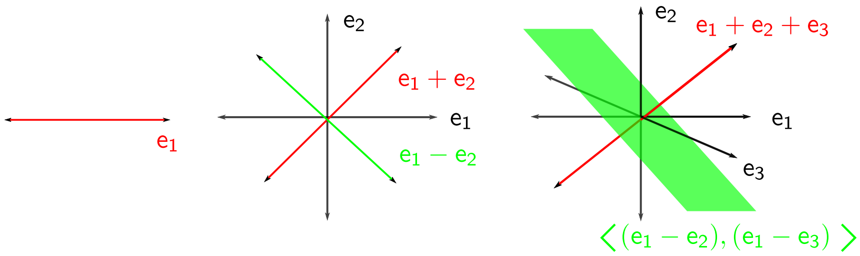
Example (The Permutation Representation)

Consider the permutation representation

$$V_n = \mathbb{Q}^n = \langle e_1, \dots, e_n \rangle.$$

For each n, V_n decomposes into two irreducibles:

$$\mathbb{Q}^n = \Big\{ a(e_1 + e_2 + \dots + e_n) \Big\} \oplus \Big\{ a_1 e_1 + \dots + a_n e_n \;\Big|\; \sum a_i = 0 \Big\}$$



An Example: The Permutation Representation

Some properties of the permutation representation

- The decomposition into irreducibles 'looks the same' for every n.

$$\mathbb{Q}^n = \Big\{ a(e_1 + e_2 + \ldots + e_n) \Big\} \oplus \Big\{ a_1 e_1 + \ldots + a_n e_n \Big| \sum a_i = 0 \Big\}$$

- The dimension of V_n grows polynomially in n

$$\dim(V_n) = n$$

- The characters χ_n of V_n have a 'nice' global description

$$\chi_n(\sigma) = \#\text{1–cycles of } \sigma \qquad \text{for all } \sigma \in S_n, \text{ for all } n.$$

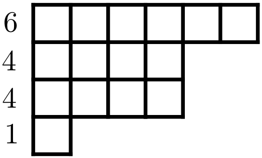
Some Representation Theory

Some facts about S_n–representations over $\mathbb{Q}$

- Every S_n–representation decomposes uniquely as a sum of irreducibles.

$$\lambda = (6, 4, 4, 1)$$

- Irreducibles are indexed by partitions λ of *n*, depicted by *Young diagrams*.



Obstacle

How can we compare irreducibles for different values of n?

Solution

Two irreducibles are "the same" if only the top rows of their Young diagrams differ.

Example (The Permutation Representation $V_n = \mathbb{Q}^n$)

$$\mathbb{Q}^1 = V_{\ydiagram{1}}$$

$$\mathbb{Q}^2 = V_{\ydiagram{2}} \oplus V_{\ydiagram{1,1}}$$

$$\mathbb{Q}^3 = V_{\ydiagram{3}} \oplus V_{\ydiagram{2,1}}$$

$$\mathbb{Q}^4 = V_{\ydiagram{4}} \oplus V_{\ydiagram{3,1}}$$

$$\mathbb{Q}^5 = V_{\ydiagram{5}} \oplus V_{\ydiagram{4,1}}$$

The Definition of an FI–module

Definition (Church–Ellenberg–Farb) (The Category FI)

Denote by FI the category of Finite sets with Injective maps

$$\{1\} \hookrightarrow \{1,2\} \hookrightarrow \{1,2,3\} \hookrightarrow \{1,2,3,4\} \hookrightarrow \{1,2,3,4,5\} \hookrightarrow$$

$$\circlearrowright \quad \circlearrowright \quad \circlearrowright \quad \circlearrowright \quad \circlearrowright$$

$$S_1 \qquad S_2 \qquad S_3 \qquad S_4 \qquad S_5$$

The Definition of an FI–module

Definition (Church–Ellenberg–Farb) (FI–Modules)

A (rational) *FI–module* is a functor

$$V \ : \ \mathrm{FI} \to \mathbb{Q}\text{-Vect}$$

$$\begin{array}{ccccccccccc}
\{1\} & \hookrightarrow & \{1,2\} & \hookrightarrow & \{1,2,3\} & \hookrightarrow & \{1,2,3,4\} & \hookrightarrow & \{1,2,3,4,5\} & \hookrightarrow & \\
\circlearrowright & & \circlearrowright & & \circlearrowright & & \circlearrowright & & \circlearrowright & & \\
S_1 & & S_2 & & S_3 & & S_4 & & S_5 & &
\end{array}$$

$$\Big\downarrow V$$

$$\begin{array}{ccccccccccc}
V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & V_4 & \longrightarrow & V_5 & \longrightarrow & \\
\circlearrowright & & \circlearrowright & & \circlearrowright & & \circlearrowright & & \circlearrowright & & \\
S_1 & & S_2 & & S_3 & & S_4 & & S_5 & &
\end{array}$$

Finite Generation of FI–Modules

Definition (Generation)

If V is an FI–module, and $S \subseteq \coprod_n V_n$, then the *FI–module generated by S* is the smallest sub–FI–module containing the elements of S.

Definition (Finite Generation)

An FI–module is *finitely generated* if it has a finite generating set.

Example (The Permutation Representation $V_n = \mathbb{Q}^n$)

The permutation representation $V_n = \mathbb{Q}^n = \langle e_1, \ldots, e_n \rangle$ is generated by $e_1 \in V_1$.

Consequences of Finite Generation

Theorem (Church–Ellenberg–Farb)

Let V be a finitely-generated FI–module. Then for $n >> 1$

- *The decomposition into irreducible S_n–representations stabilizes.*
- *$\dim(V_n)$ is polynomial in n*
- *The characters χ_n of V_n are given by a (unique) polynomial in the variables X_r*

$$X_r(\sigma) = \#r\text{–cycles of } \sigma \qquad \text{for all } \sigma \in S_n, \text{ for all } n.$$

Any sub–FI–module of V also has these properties.

We call the sequence $\{V_n\}_n$ *uniformly representation stable*.

Some Representation Stable Cohomology Sequences

(Church–Farb)	$\{H^k(P_n;\mathbb{Q})\}_n$	**The pure braid group**
(Jimenez-Rolland)	$\{H^k(\mathrm{PMod}(\Sigma^n_{g,r});\mathbb{Q})\}_n$	**The pure MCG of an *n*-puncture surface** $\Sigma^n_{g,r}$
(Church)	$\{H^k(\mathrm{PConf}_n(M);\mathbb{Q})\}_n$	**Ordered configuration space of a manifold** M
(Putman)	$\{H^k(\mathrm{PMod}^n(M);\mathbb{F})\}_n$	**The pure MCG of an *n*-puncture manifold**
(Putman)	Eg, $\{H^k(\mathrm{SL}_n(\mathbb{Z},\ell);\mathbb{F})\}_n$	**Certain congruence subgroups**
(Wilson)	$\{H^k(P\Sigma_n;\mathbb{Q})\}_n$	**The pure symmetric automorphism group** $P\Sigma_n$ **of the free group**

Open Question

Problem

Compute the characters, and the stable decompositions into irreducibles, in the above examples.

My Research

My current project

To develop a unified “FI–module theory” for the three families of classical Weyl groups.

References

Further Reading

T Church, B Farb.
Representation theory and homological stability, preprint, 2010.

T Church, J Ellenberg, B Farb.
FI–modules: A new approach to stability for S_n–representations, preprint, 2012.

The End

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