
Goal: Constructing the spectral sequence associated to a filtered space \( \emptyset \subseteq x_0 \subseteq x_1 \subseteq \cdots \subseteq x_n = X \).

We will assume:
1. finite filtration
2. spaces have homology groups of finite dimension/rank.

Intuition: Starting from the relative groups \( H_n (X_0, X_{n-1}) \), we will construct "successive approximations" to the homology \( H_n (X) \).

Preliminaries: Exact Couples.

Defn: An exact couple is an exact sequence of abelian groups of the form:

\[
\begin{array}{ccc}
A & \xrightarrow{i} & A \\
\downarrow{k} & & \downarrow{j} \\
E & & \\
\end{array}
\]

with \( i, j, k \) group homomorphisms.

Define map \( d := j \circ k : E \rightarrow E \).

Claim \( d^2 = 0 \).

Proof:
\[
d^2 = (j \circ k) \circ (j \circ k) = (j) \circ (k \circ j) \circ (k) = 0
\]

by exactness.

We will give a procedure for extending \((E, d)\) to a spectral sequence.

Make the following definitions:

\[
E^2 = \frac{\ker d}{\text{im} d} \quad A^2 = \text{i}(A)
\]
\[ i^2 : A^2 \rightarrow A^2 \quad i^2 = i(\cdot) \]
\[ k^2 : E^2 \rightarrow A^2 \quad k^2 \text{ is the induced map on } E^2 \text{ by } k. \]
\[ j^2 : A^2 \rightarrow E^2 \]
\[ j^2 : i(a) \rightarrow [j^2(a)] \]

*NB* the superscripts are indices, and not exponents.

It is straightforward to check that these maps are well-defined.

These definitions give the derived couple:

\[ A^2 = i(A) \xrightarrow{i^2} A^2 = i(A) \]

\[ k^2 \]

\[ E^2 = \frac{\text{ker}(d)}{\text{im}(d)} \]

Claim: The derived couple is an exact couple.

PF: A diagram chase.

By iterating this construction, we get an exact couple

\[ A^r \xrightarrow{i^r} A^r \quad \text{for all } r \in \mathbb{N} \]

\[ k^r \]

\[ E^r \xrightarrow{j^r} E^r \]

with differentials

\[ d^r := j^r k^r \]

The sequence \( \{E^r, d^r\} \) is the data of the spectral sequence.
The spectral sequence of a filtered complex.

Given a finite filtration of topological spaces:

\[ \emptyset \subset X_0 \subset X_1 \subset \ldots \subset X_n = X \]

for each pair \((X_p, X_{p-1})\) we have a long exact sequence:

\[ \rightarrow H_n(X_{p-1}) \stackrel{i}{\rightarrow} H_n(X_p) \stackrel{j}{\rightarrow} H_n(X_p, X_{p-1}) \stackrel{k}{\rightarrow} H_{n-1}(X_{p-1}) \rightarrow \]

which we can condense:

\[ \begin{array}{ccc}
H_*(X_{p-1}) & \stackrel{i}{\rightarrow} & H_*(X_p) \\
 & k \uparrow & j \downarrow \\
 & H_*(X_p, X_{p-1}) & \\
\end{array} \]

which gives the following complex:

\[ 0 \rightarrow H_*(X_0) \rightarrow H_*(X_1) \rightarrow H_*(X_2) \rightarrow H_*(X_3) \rightarrow \ldots \]

\[ \begin{array}{ccc}
i_1 & & i_2 \\
\downarrow k_1 & & \downarrow k_2 \\
& H_*(X_1, X_0) & \\
\end{array} \]

which is exact around each triangle.

If we condense this further:

\[ A = \bigoplus_p H_*(X_p) \quad E = \bigoplus_p H_*(X_p, X_{p-1}) \]

then the result is our exact couple

\[ \begin{array}{ccc}
A & \stackrel{i}{\rightarrow} & A \\
& k \uparrow & \downarrow j \\
E & \rightarrow & \end{array} \]

Degrees: Take \(q = n - p\).

\[ i: (p-1, q) \rightarrow (p, q-1) \quad \text{degree} \ (1, -1) \]

\[ j: (p, q) \rightarrow (p, q) \quad \text{degree} \ (0, 0) \]

\[ k: (p, q) \rightarrow (p-1, q) \quad \text{degree} \ (-1, 0) \]
The $E^1$ page: spectral sequence of a filtered complex $X_p$
Our differentials have the form:

\[ d^n = j \circ k = j_{p+1} \circ k_p \]
\[ d^2 = j \circ i \circ k = j_{p-2} \circ i_{p-1} \circ k_p \]
\[ d^r = j \circ i^{-(r-1)} \circ k = j_{p-r} \circ i^{-(r+1)} \circ k_p. \]

Note: Abuse of notation — the map \( i \) may not be invertible, but the claim is that these composites are well-defined.

**Evolution of the sequence in an easy case:**

Consider the filtration of spaces \( 0 \leq X_0 \leq X_1 \leq X \)

\[ A_1: 0 \rightarrow H_*(X_0) \rightarrow i_* H_*(X_1) \rightarrow i_* H_*(X) \rightarrow H_*(X) \rightarrow 0 \]

Composing the image of this map, \( i_* H_*(X_1) \), with an isomorphism gives the inclusion of the image.

\[ A_2: 0 \rightarrow i_* H_*(X_0) \rightarrow i_* H_*(X_1) \hookrightarrow H_*(X) \rightarrow H_*(X) \rightarrow 0 \]

\[ \cup_i \quad H_*(X) \rightarrow H_*(X) \quad \text{inclusion} \]

\[ A_3: 0 \rightarrow i^2_* H_*(X_0) \rightarrow i^2_* H_*(X_1) \rightarrow i^2_* H_*(X) \rightarrow H_*(X) \rightarrow 0 \]

\[ \cup_i \quad H_*(X) \rightarrow H_*(X) \quad \text{inclusion} \]

By \( A_3 \), all maps are injective.

Since the filtration is finite, the sequence eventually "runs into" isomorphisms, and stabilizes at a filtration of \( H_*(X) \) by the images of the absolute groups \( H_*(X_p) \).
Since $i^3$ is injective, by exactness, $k^3 \equiv 0$ and $j^3$ surjects thus each $E^3_p = i^2 H^*_p(X_p) / i^2 H^*_p(X_{p-1})$ [by exactness at $i^2 H^*_p(X_p)$]

In general, we find $E^\infty_{p,q} = F^p_n / F^p_{n-1}$ ($n = p+q$) for filtration $0 \leq F^0_0 \leq \cdots \leq F^n_n = H^n(X)$ with $F^p_n = \text{Im} \{ H^n(x) \xrightarrow{i} H^n(X_p) \}$

There is an analogous construction in cohomology with maps reversed, in this case $E^p_{q,p} = F^p_n / F^p_{n-1}$ ($n = p+q$) for filtration $0 \leq F^0_0 \leq \cdots \leq F^n_n = H^n(X)$ with $F^p_n = \text{Ker} \{ H^n(x) \xrightarrow{i} H^n(X_p) \}$
What is happening at each step?

\[ d^2 \quad \text{induces} \quad d^2 \]

\[
\begin{align*}
&H_* (X_0) \rightarrow H_* (X_1) \\
&H_* (X_1, X_0) \rightarrow H_* (X_2, X_1) \\
&H_* (X_2, X_1) \rightarrow H_* (X_3, X_2) \\
&H_* (X_3, X_2) \rightarrow \ldots
\end{align*}
\]

\[ d^r = j \circ i^{-(r-1)} \circ k \]

Goal: Identify elements of \( H_* (X_p, X_{p-1}) \) that lift (through \( j \)) to \( H_* (X_p) \), and survive (through \( i^{p-p} \)) to \( H_* (X) \).

Given \( x \in H_* (X_p, X_{p-1}) \)

- If \( x \not\in \ker (d^r) \) for some \( r \),
  then \( x \not\in \ker (k) = \text{im} (j) \), and \( x \) does not lift to \( H_* (X_p) \).
- If \( x \in \text{im} (d^r) \) for some \( r \),
  \( x = j \circ i^{-(r-1)} \circ k (y) \),
  then \( x \) lifts through \( j \) to \( i^{-(r-1)} \circ k (y) \),
  which maps through \( i^{r-1} \) to \( k (y) \) \( \in \text{im} (k) = \ker (i) \).
  The lift of \( x \) does not represent an element of \( H_* (X) \).
- Otherwise - \( x \) survives the spectral sequence to \( E^\infty \)

**Exercise**

Verify the following: If \( X \) is a CW-complex with \( p \)-skeletons \( X_p \)

\[ E_{p,q}^1 = H_{p+q} (X_p, X_{p-1}) = \begin{cases} 
\bigoplus \mathbb{Z} & q = 0 \\
0 & \text{otherwise}
\end{cases} \]

- \( d^1 \) is the usual differential from cellular homology
- The spectral sequence degenerates at \( E^2 = E^\infty \),
  all higher differentials are 0.
Constructing the Leray-Serre spectral sequence.

The Leray-Serre spectral sequence arises as a special case of the spectral sequence of a filtered complex.

Given a fibration \( F \rightarrow X \xrightarrow{\pi} B \)

Suppose \( B \) is a CW-complex (or else pull back the fibration to a CW approximation of \( B \))

- Filter \( B \) by its \( p \)-skeleta \( B^p \)
- Filter \( X \) by their inverse images \( \pi^{-1}(B^p) = X^p \)

The Leray-Serre spectral sequence is the sequence associated to the filtration \( X^p \) of \( X \).