what is a spectral sequence?

A spectral sequence is a computational tool; they are more complex analogues of long exact sequences.

Eg. Just as there is a LES of a pair \((X,A)\) in homology/cohomology, there is a spectral sequence associated to a filtration of subspaces \(\emptyset \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X\).

Eg. Analogous to the Mayer-Vietoris LES, there is a spectral sequence associated to an open cover \(\{U_i\}\) of \(X\).

The structure of a spectral sequence.

A spectral sequence is a "book" consisting of a sequence of pages (or sheets), denoted \(E^r\) (homology) or \(E^r\) (cohomology) with \(r \in \mathbb{N}\).

Each page has

- A 2D array of groups (or rings, or algebras)
  
  \[ E^r_{p,q}, \quad (p,q) \in \mathbb{Z}^2 \]

- A map \(d^r: E^r \rightarrow E^r\) satisfying \((d^r)^2 = 0\), called the differential.

Caution: \(r\) is an index, not an exponent.

The differentials \(d^r\) give \(E^r\) the structure of a chain complex.
The page $E^{r+1}$ is determined by the homology of $E^r$
with respect to $d^r$
(though the differentials $d^{r+1}$ are not determined
by $E^r$, $d^r$).

Our main example will be the Leray-Serre Spectral sequence.

In the homology version:

- $E^r_{p,q}$ is non zero only for $p,q \geq 0$.
- $d^r_{p,q} : E^r_{p,q} \rightarrow E^r_{p-r, q+r-1}$

![Diagram of the Spectral Sequence](image)

- Dots are abelian groups $E^r_{p,q}$.
- Differentials $d^r$.
- $d^r_{p,q} : E^r_{p,q} \rightarrow E^{r+1}_{p,q}$

Each abelian gp $E^2_{p,q}$ is a subquotient of
the group $E^r_{p,q}$.

Convergence:

Defn: we say a spectral sequence $\{E^r_{p,q}, d^r\}$ converges to a
limit (denoted $E^{\infty}_{p,q}$) if, for some sufficiently large $N$
(depending on $p,q$), all differentials leaving and entering
$E^r_{p,q}$ are $0$ for $r \geq N$. 
In this case, the homology groups are isomorphic for \( r \geq N \):
\[
E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \ldots
\]

and these stable groups are called \( E_r^{\infty} \).

**Defn.** We say a spectral sequence degenerates at page \( N \) if \( d_r \equiv 0 \) for all \( r \geq N \); in this case \( E^N = E^{\infty} \).

**Note:** Since the Leray-Serre spectral sequence is 0 outside of the first quadrant, it must converge: For each \( (p,q) \), for \( r \) sufficiently large, the outgoing differential must eventually land outside the first quadrant, and the incoming differential must eventually originate outside the first quadrant.

**Eq.** \( E^3_{1,1} = E^{\infty}_{1,1} \)

**Notation:** We write
\[
E_r^{p,q} \Rightarrow E^{\infty}_{p,q}.
\]

The Leray-Serre spectral sequence. (Homology version)

**Thm.** Let \( F \rightarrow X \rightarrow B \) be a fibration; \( B \) path-connected.

Suppose \( \pi_1(B) \) acts trivially on \( H_x(F,G) \).

Then there is a spectral sequence \( \{ E_r^{p,q}, d^r \} \) such that:

- \( d^r : E_r^{p,q} \rightarrow E_r^{p-r, q+r-1} \), \( E_r^{p,q} = \ker(d^r) / \text{im}(d^r) \) at \( E_r^{p,q} \).
- \( E^2_{p,q} = H_p(B ; H_q(F,G)) \). For \( G \) a field, \( E^{2,q}_p = H_p(B) \otimes_G H_q(F) \).
- Stable terms \( E^{\infty}_{p,q} \) are isomorphic to successive quotients \( F_n^p / F_{n-1}^p \) in a filtration of \( H_n(X; G) \): (\( n = p+q \))

\[
0 \leq F_0^p \leq F_1^p \leq \ldots \leq F_n^p = H_n(X; G).
\]
Remarks (Action of $\pi_1(B)$ on $H^*(F;G)$).

Recall: Given a fibre $\pi^{-1}(b)$, a point $x \in \pi^{-1}(b)$ and a loop in $B$ based at $b$,
we can lift the loop to a path from $x$ to some $x' \in \pi^{-1}(b)$.
Thus the loop defines a map $\pi^{-1}(b) \to \pi^{-1}(b)$.

Given an element $[\gamma] \in \pi_1(B, b)$ with rep. $x$, there is a
map $\gamma: \pi^{-1}(b) \to \pi^{-1}(b)$; since the fibration satisfies
the homotopy lifting property, there is an induced action
of $\pi_1(B, b)$ on $F \wedge \pi^{-1}(b)$ well-defined up to homotopy,
and hence an induced action $\pi_1(B) \wedge H^*(F;G)$.

We may think of the condition that this action is trivial as an
"orientability condition", giving a consistent identification
of the groups $H^n(F;G)$ with the cohomology of each fibre.

If the action $\pi_1(B) \wedge H^*(F;G)$ is nontrivial, then the
statement of the Leray-Serre spectral sequence can be
formulated using homology with "twisted coefficients"
the $\pi_1(B)$-module $H^*(F;G)$.

Remark (Cohomology Version)

There is a cohomology version of the Leray-Serre spectral sequence
of a fibration $F \to X \to B$. (under same assumptions)

with $E_2^{p,q} = H^p(B; Hq(F;G)) \Rightarrow H^{p+q}(X;G)$

(again converging to successive quotients $E_\infty^{p,n-p} = F^n / F_{n+1}^n$
of $0 \leq F_0^n \leq \cdots \leq F_n^n = H^n(X;G)$)

The differentials' directions are reversed: $d_r : E_r^{p,q} \to E_r^{p+r,q-r}$
and $E_r^{p,q} = \text{Ker}(d_r) / \text{Im}(d_r)$ at $E_r^{p,q}$.
Remark (the $E^2$ page)

At first encounter, the structure of the $E^2$ page may appear strange.

A remark, though: if we take $X = F \times B$, and we take the (co)homological version of the Leray-Serre spectral sequence over a field, then the sequence degenerates at $E_2$ and gives the Kunneth formula.

Remark (Recovering $H_*(X)$)

The sequence converges to successive quotients in a filtration of $H_*(X)$.

In general, this only determines $H_*(X)$ "up to extensions".

As an easy example, consider the short exact sequences:

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow ? \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

The extension $?$ could be either $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

However, if we work with coefficients in a field, or if we have a spectral sequence converging to free $\mathbb{Z}$-modules, then the limit fully determines $H_*(X) = \bigoplus P E_{p,q} \infty$. 
Example: Hopf Fibration \( S^3 \rightarrow S^7 \rightarrow S^4 \)

\[
H_p(B; \mathbb{Z}) = H_p(S^4; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & p = 0,4 \\ 0, & \text{otherwise} \end{cases}
\]

\[
H_q(F; \mathbb{Z}) = H_q(S^3; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0,3 \\ 0, & \text{otherwise} \end{cases}
\]

NB: \( S^4 \) is simply connected, so monodromy is trivial.

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Thus \( d_4: E_0^{2,3} \rightarrow E_4^{4,0} \), the only possible nonzero differential, must be an isomorphism killing off \( E_0^{2,3} \) and \( E_4^{4,0} \).

The sequence degenerates at \( E_5^{5} \).

NB: We can use the structure of this spectral sequence to exclude some possibilities for fibrations of the form:

\[
S^F \rightarrow S^X \rightarrow S^B
\]
Example: Using the Leray-Serre spectral sequence to compute $H_*(\mathbb{C}P^{\infty}; \mathbb{Z})$ from $H_*(S^2; \mathbb{Z})$ and $H_*(S^\infty; \mathbb{Z})$

we have a fibration $S^1 \to S^\infty \to \mathbb{C}P^{\infty}$

so $E_2^{p,q} = H_p(\mathbb{C}P^{\infty}; H_q(S^2; \mathbb{Z}))$

\[
= \begin{cases} 
H_p(\mathbb{C}P^{\infty}; \mathbb{Z}), & q=0,1 \\
0 & \text{otherwise}
\end{cases}
\]

Again, $\mathbb{C}P^{\infty}$ is simply connected.

sequence converges to

$E_\infty^{0,0} = H_0(S^\infty; \mathbb{Z}) = \mathbb{Z}$

and $E_\infty^{p,q} = 0$ for all other $p, q$.

Since $S^\infty$ is contractible.

* The only possible nonzero differentiable is $d^2$, so $E^3 = E_\infty$.

There are no nonzero differentials to kill this term, so we must have $H_1(\mathbb{C}P^{\infty}) = 0$.

Since $E_2^{3,0}$ must die in the limit, we conclude $H_3(\mathbb{C}P^{\infty}) = 0$.

Iterating this argument, we conclude inductively that $H_{2n}(\mathbb{C}P^{\infty}) = 0$ if $n$ is odd.

Since $E_2^{2,0} = H_2(\mathbb{C}P^{\infty})$ must die in the limit, as $E_2^{3,0} = H_2(\mathbb{C}P^{\infty})$, we must have $H_2(\mathbb{C}P^{\infty}) = \mathbb{Z}$, and $d_2^{2,0} : H_1(\mathbb{C}P^{\infty}) \to H_0(\mathbb{C}P^{\infty})$.

Then $E_2^{2,1} \neq H_1(\mathbb{C}P^{\infty}) \neq \mathbb{Z}$ must be killed off by $H_4(\mathbb{C}P^{\infty})$ and continuing inductively $H_n(\mathbb{C}P^{\infty}) = \mathbb{Z}$ for all $n$ even.