Path integrals and optical tomography

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Abstract. We make use of the path integral representation of the diffusion Green's function to derive a renormalized perturbation expansion for the propagation of diffuse waves in an inhomogeneous medium. The first term in the expansion coincides with the Rytov approximation. The higher order terms can be used to develop a direct reconstruction method for time-resolved optical tomography.

1. Introduction

Optical tomography is an emerging biomedical imaging modality that uses diffuse light as a probe of tissue structure and function [1, 2]. In a typical experiment, a highly-scattering medium is illuminated by a point source and the light that is transmitted through the medium is collected by an optical fiber. The mathematical problem that is considered is to reconstruct the optical properties of the medium from boundary measurements. The first generation of optical tomography systems were designed to measure the time-resolved intensity of light transmitted through a medium of interest. More recently, noncontact continuous-wave imaging systems have been introduced, wherein a scanned beam and a lens-coupled CCD is employed to replace the illumination and detection fiber-optics of earlier instruments. Using such a noncontact method, extremely large data sets of approximately $10^8$ measurements can readily be obtained.

The development of image reconstruction algorithms in optical tomography has paralleled the above advances in instrument design. Nonlinear optimization methods were utilized in early optical tomography systems. Later, fast image reconstruction algorithms [5, 6, 11] were introduced in order to handle the large data sets of noncontact instruments. Such algorithms have well understood convergence, error and stability [7, 8] and have been tested in experiments that demonstrate a substantial improvement in image quality [4, 13].

In this paper we revisit some aspects of the theory of time-resolved optical tomography. The motivation is two-fold. (i) Large data sets derived from time-resolved noncontact imaging systems are now available [12]. This development makes possible the simultaneous reconstruction of the absorption and scattering
coefficients, which is impossible using continuous-wave measurements [1]. (ii) Fast image reconstruction algorithms have been principally formulated and analyzed in the continuous-wave setting. Evidently, it would be of some interest to extend these results to the time-dependent case. Our results may be summarized as follows. We reformulate the scattering theory for diffuse waves in the language of Feynman path integrals. This approach focuses on the role of photon paths and is complementary to the usual description of light propagation in terms of diffusion equations. We then develop the integral equations of scattering theory and make contact with the Rytov approximation. It is shown that the path integral approach may be interpreted as a generalization of the Radon transform of x-ray computed tomography. Finally, we indicate some applications to the inverse problem of optical tomography.

2. Path Integrals and Photon Diffusion

We begin by recalling some facts about the propagation of diffuse light in a random medium. We assume that the wavelength is much smaller that the transport mean free path $l^*$, which is itself much smaller than the system size. The quantity $l^*$ may be interpreted as the distance a photon travels before its direction is randomized. The path of a photon may be thus be regarded as a random walk with step size $l^*$. In this picture, in the absence of absorption, a photon path $r(t)$ obeys the stochastic differential equation

$$\frac{dr(t)}{dt} = \eta(t),$$

where $\eta(t)$ is the Gaussian white-noise process with correlation functions

$$\langle \eta(t) \rangle = 0,$$

$$\langle \eta_i(t) \eta_j(t') \rangle = 2D \delta_{ij} \delta(t - t').$$

Here the diffusion coefficient $D = \frac{1}{3}(c/n)l^*$, where $c/n$ is the speed of light in the medium. The expectation $\langle \cdots \rangle$ is defined with respect to the probability distribution functional of the white noise as

$$\langle \cdots \rangle = \int \prod_t d^3 \eta(t) \exp \left[ -\frac{1}{4D} \int_0^t dt' \eta^2(t') \right] \cdots ,$$

where an overall normalization has been omitted.

Eq. (1) defines the time evolution of the path of a single photon. The corresponding probability density $p(r, t) = \langle \delta(r - r(t)) \rangle$ obeys the diffusion equation

$$\frac{\partial}{\partial t} p(r, t) = D \nabla^2 p(r, t),$$

as shown in the appendix. The probability distribution functional for a photon path may be obtained from (1) and (3). The probability weight $W[r(t)]$ assigned to a path $r(t)$ is thus seen to be

$$W[r(t)] \propto \exp \left[ -\frac{1}{4D} \int_0^t dt' \dot{r}^2(t') \right].$$

The corresponding probability measure $d\mu$ on the space of paths is given by

$$d\mu = \exp \left[ -\frac{1}{4D} \int_0^t \dot{r}^2(t') dt' \right] \prod_t d^3 r(t).$$
The diffusion Green’s function $G(r_1, r_2; t)$ is the conditional probability of finding a photon at $r_2$ at time $t$ given that it began at $r_1$ at $t = 0$. Integrating over paths with weight given by (5), we obtain

$$G(r_1, r_2; t) = \int d\mu \delta(r_1 - r(0))\delta(r_2 - r(t))$$

which defines the measure $Dr$. The Green’s function is thus expressed as a sum over paths, where each path is weighted by the exponential factor in (5). The path with highest probability weight minimizes the argument of the exponential in (6) and corresponds to the line connecting $r_1$ and $r_2$. See Figure 1.

We denote by $d\mu[r_1, 0; r_2, t]$ the probability measure on the space of photon paths which begin at $r_1$ and end at $r_2$ at time $t$. It is defined by the expression

$$d\mu[r_1, 0; r_2, t] = \frac{\delta(r_1 - r(0))\delta(r_2 - r(t))}{\int d\mu \delta(r_1 - r(0))\delta(r_2 - r(t))} d\mu .$$

Using the fact that the denominator in (8) is the Green’s function $G(r_1, r_2; t)$, we see that $d\mu[r_1, 0; r_2, t]$ becomes

$$d\mu[r_1, 0; r_2, t] = \frac{1}{G(r_1, r_2; t)} \delta(r_1 - r(0))\delta(r_2 - r(t))d\mu .$$

The extension of the above results to the case of photon diffusion with absorption is relatively straightforward. The probability weight must be modified and becomes

$$W[r(t)] \propto \exp\left[ - \int_0^t dt' \left( \frac{1}{4D} \mathbf{r}^2(t') + \alpha(r(t')) \right) \right] ,$$
where \( \alpha \) is the absorption coefficient. The path integral representation of \( G(r_1, r_2; t) \) is then given by

\[
G(r_1, r_2; t) = \int Dr \exp \left[ - \int_0^t dt' \left( \frac{1}{4D} \vec{r}^2(t') + \alpha(r(t')) \right) \right].
\]

The diffusion equation corresponding to (11) is of the form

\[
\frac{\partial}{\partial t} p(r, t) = D \nabla^2 p(r, t) - \alpha(r)p(r, t).
\]

Note that \( p \) is proportional to the energy density of the diffuse wave and is not conserved.

### 3. Integral Equations

Consider an experiment in which an optical pulse from a point source at \( r_1 \) is registered by a point detector at \( r_2 \) at time \( t \). The transmission coefficient \( T(r_1, r_2, t) \) is defined as the transmitted intensity normalized by the intensity that would be measured in the absence of absorption. Using (11) we find that

\[
T(r_1, r_2, t) = \int d\mu[r_1, 0; r_2, t] \exp \left[ - \int_0^t dt' \alpha(r(t')) \right].
\]

The above result may be used to generate a perturbation expansion for \( T \) in powers of \( \alpha \). To proceed, we expand the exponential in (13) and obtain

\[
T = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d\mu[r_1, 0; r_2, t] \int_0^t dt_1 \cdots \int_0^t dt_n \alpha(r(t_1)) \cdots \alpha(r(t_n)).
\]

Next we introduce the correlation functions

\[
\Gamma^{(n)}(R_1, \ldots, R_n; r_1, r_2, t) = \int d\mu[r_1, 0; r_2, t] \int_0^t dt_1 \cdots \int_0^t dt_n \delta(R_1 - r(t_1)) \cdots \delta(R_n - r(t_n))
\]

and rewrite (14) as

\[
T = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^3 R_1 \cdots d^3 R_n \Gamma^{(n)}(R_1, \ldots, R_n; r_1, r_2, t) \alpha(R_1) \cdots \alpha(R_n).
\]

Eq. (16) is the Born series for diffuse waves. It describes the “multiple scattering” of a diffuse wave from inhomogeneities in the absorption \( \alpha \).

We now reexponentiate the series in (14) by making the ansatz

\[
T = \exp \left[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d^3 R_1 \cdots d^3 R_n \Gamma^{(n)}_c(R_1, \ldots, R_n; r_1, r_2, t) \alpha(R_1) \cdots \alpha(R_n) \right],
\]

where the connected correlation function \( \Gamma^{(n)}_c \) is to be determined. Evidently, \( \ln T \) is the generating functional for \( \Gamma^{(n)}_c \):

\[
\Gamma^{(n)}_c(R_1, \ldots, R_n; r_1, r_2, t) = \frac{\delta^n \ln T(r_1, r_2, t)}{\delta \alpha(R_1) \cdots \delta \alpha(R_n)} \bigg|_{\alpha=0}.
\]
Carrying out the functional differentiations, we find that $\Gamma^{(1)}_c$ and $\Gamma^{(2)}_c$ are given by

$$\Gamma^{(1)}_c = \Gamma^{(1)}(R_1; r_1, r_2, t) ,$$

$$\Gamma^{(2)}_c = \Gamma^{(2)}(R_1, R_2; r_1, r_2, t) - \Gamma^{(1)}(R_1; r_1, r_2, t)\Gamma^{(1)}(R_2; r_1, r_2, t) .$$

Eq. (17) is the renormalized Born series for diffuse waves. It is the path integral analog of the Rytov series in scattering theory.

The correlation function $\Gamma^{(1)}$ is known as the hitting density [10]. It can be seen that

$$\Gamma(r; r_1, r_2, t) = \frac{1}{G(r_1, r_2; t)} \int_0^t dt' G(r_1, r; t')G(r, r_2; t - t') .$$

We note that the hitting density in an infinite medium is given by the formula

$$\Gamma(r; r_1, r_2, t) = \frac{1}{4\pi D} \left( \frac{1}{|r - r_1|} + \frac{1}{|r - r_2|} \right) \times \exp \left[ -\frac{1}{4Dt} \left( |r - r_1| + |r - r_2| \right)^2 - (r_1 - r_2)^2 \right] .$$

Here we have used the integral representation of the Green’s function

$$G(r_1, r_2; t) = \frac{1}{(4\pi Dt)^{3/2}} \exp \left[ -\frac{1}{4Dt} (r_1 - r_2)^2 \right]$$

$$= \int \frac{dz}{2\pi i} e^{zt} \int \frac{d^3k}{(2\pi)^3} e^{iK(\vec{r}_1, \vec{r}_2)} \frac{1}{Dk^2 + z} .$$

to evaluate the above integral. Let $L = |r_1 - r_2|$ denote the source-detector separation and define the diffusion time $\tau_D = L^2/D$. In the short-time limit $t \ll \tau_D$, the hitting density is concentrated on the line connecting the source and detector where the photon’s path “hits” most frequently. This represents the dominant contribution from nearly unscattered photons. In the long-time limit $t \gg \tau_D$ the principal contribution to the hitting density is from photons with longer path lengths. See Figure 2.

If we keep only the first term in (17) and make use of (15) we obtain the integral equation

$$-\ln T(r_1, r_2, t) = \int d\mu[r_1, 0; r_2, t] \int_0^t dt' \alpha(r(t')) ,$$

which is the analog of the Rytov approximation for diffuse waves. It is readily appreciated that the above result may be interpreted as a path-integral generalization of the two-dimensional Radon transform of x-ray computed tomography [9]. Recall that the x-ray transmission coefficient $T$ is related to the attenuation coefficient $\mu$ by

$$-\ln T = \int_L \mu(r) dr ,$$

where $L$ is the line along which the x-ray beam propagates. Note that in (25) the integration is along photon paths weighted by $d\mu$, while in (26) it is along a fixed line.
Figure 2. Contour plots of \( \Gamma^{(1)} \) in the source-detector plane with \( L = 8 \) cm and \( D = 1 \) cm\(^2\) ns\(^{-1}\). The left panel shows the case \( t \ll \tau_D \) and the right panel the case \( t \gg \tau_D \).

If we exponentiate (1) and expand the result to first order in \( \alpha \), we find that

\[
T(r_1, r_2, t) = 1 - \int d\mu(r_1, 0; r_2, t) \int_0^t dt' \alpha(r(t'))
\]

\[
= 1 - \int d^3 r \Gamma^{(1)}(r; r_1, r_2, t) \alpha(r)
\]

(27)

where we have used (17). This result, which corresponds to the first Born approximation for diffuse waves, was first derived in [3].

4. Conclusions

We have reexamined the theory of the forward problem in time-resolved optical tomography. Our main result, Eq. (17), is a renormalized perturbation theory for diffuse waves, which we have derived by making use of the language of Feynman path integrals. The first term in the series corresponds to the Rytov approximation while the higher order terms provide higher order corrections.

The series (17) can be used to develop a solution to the inverse problem. To see this, we rewrite (17) as a power series in tensor powers of \( \alpha \) of the form

\[
\phi = K_1 \alpha + K_2 \alpha \otimes \alpha + K_3 \alpha \otimes \alpha \otimes \alpha + \cdots
\]

(28)

where \( \phi = -\ln T \) and \( K_n = (-1)^n/n! \Gamma^{(n)}_c \). The inverse problem is to determine \( \alpha \) everywhere inside of a bounded domain from measurements of \( \phi \) on its boundary. Following [5, 7], we express \( \alpha \) as a power series in tensor powers of \( \phi \):

\[
\alpha = K_1 \phi + K_2 \phi \otimes \phi + K_3 \phi \otimes \phi \otimes \phi + \cdots
\]

(29)
It can be seen that the operators $K_n$ are given by

\begin{align}
K_1 &= K_1^+, \\
K_2 &= -K_1 K_2 K_1 \otimes K_1 , \\
K_3 &= -(K_2 K_1 \otimes K_2 + K_2 K_1 \otimes K_1) K_1 \otimes K_1 \otimes K_1 , \\
K_j &= -\left(\sum_{m=1}^{j-1} K_m \sum_{i_i + \cdots + i_m = j} K_{i_1} \otimes \cdots \otimes K_{i_m}\right) K_1 \otimes \cdots \otimes K_1 .
\end{align}

Here $K_1^+$ is the pseudoinverse of the operator $K_1$, which can be constructed following the methods of [6, 11]. Eq. (29) is the inverse of the renormalized Born series. It is the starting point for the development of a direction inversion method.

The convergence and stability of the inverse Born series has been characterized for the case of continuous-wave optical tomography [7]. In addition, numerical studies have been performed which validate the theory. Evidently, it would be of interest to pursue the analogous questions in the time-dependent case.

**Appendix**

We show here that the distribution of photon paths of the diffusion process satisfies the diffusion equation. To proceed we take the time derivative of $p(r, t) = \langle \delta(r - r(t)) \rangle$ and use the equations of motion (1) to obtain

\[ \frac{\partial}{\partial t} p(r, t) = \left\langle \sum_i \eta_i(t) \frac{\delta}{\delta r_i(t)} \delta(r - r(t)) \right\rangle . \]

In (34) the symmetry of the $\delta$-function allows the replacement of the functional derivative by a partial derivative which can then be taken out of the average. Eq.(34) thus becomes

\[ \frac{\partial}{\partial t} p(r, t) = -\sum_i \frac{\partial}{\partial r_i} \left\langle \eta_i(t) \delta(r - r(t)) \right\rangle . \]

All that remains is the evaluation of the expectation in (35). Using the equations of motion and the chain rule we find that

\[ \left\langle \eta_i(t) \delta(r - r(t)) \right\rangle = 2D \left\langle \frac{\delta}{\delta \eta_i(t)} \delta(r - r(t)) \right\rangle \]

\[ = 2D \left\langle \sum_j \frac{\delta r_j(t)}{\delta \eta_i(t)} \frac{\delta}{\delta r_j(t)} \delta(r - r(t)) \right\rangle . \]

Finally, we calculate the remaining functional derivative in (35) by integrating the equations of motion:

\[ r_j(t) = r_j(0) + \int_0^t \eta_j(t') dt' , \]

and then differentiating the result with respect to $\eta_i(t)$ to obtain

\[ \frac{\delta r_j(t)}{\delta \eta_i(t)} = \frac{1}{2} \delta_{ij} . \]
Using these results and again replacing the functional derivative by an ordinary derivative we get

\[
\langle \eta_i(t)\delta(\mathbf{r} - \mathbf{r}(t)) \rangle = -D \frac{\partial}{\partial r_i} p(\mathbf{r}, t),
\]

which shows that \(p(\mathbf{r}, t)\) satisfies the diffusion equation (4).

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References