COX RINGS AND PSEUDEFFECTIVE CONES OF
PROJECTIVIZED TORIC VECTOR BUNDLES

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Abstract. We study projectivizations of a special class of toric vector bundles, whose associated Klyachko filtrations are particularly simple. For these projectivized bundles, we give generators for the cone of effective divisors and a presentation of the Cox ring as a polynomial algebra over the Cox ring of a blowup of projective space at finitely many points. These constructions yield many new examples of Mori dream spaces, as well as examples where the pseudoeffective cone is not polyhedral. In particular, we show that some projectivized toric vector bundles are not Mori dream spaces.

1. Introduction

Projectivizations of equivariant vector bundles over complete toric varieties are a large class of rational varieties that have interesting moduli and share some of the pleasant properties of toric varieties and other Mori dream spaces. Hering, Mustaţă, and Payne showed that their cones of effective curves are polyhedral and asked whether their Cox rings are indeed finitely generated [HMP10]. There are multiple proofs of an affirmative answer for rank two bundles [Kno93, HS10, Gon10].

Here, we apply general results of Hausen and Süss on Cox rings for varieties with torus actions to give a presentation of the Cox ring for certain projectivized vector bundles as a polynomial algebra over the Cox ring of the blowup of projective space at a finite set of points. The question of finite generation for these Cox rings of blowups is completely understood when the points are in very general position, through work of Mukai [Muk04] and Castravet and Tevelev [CT06] in connection with Hilbert’s fourteenth problem.

Let $X$ be the smooth projective toric variety of dimension $d$, over an uncountable field $k$, corresponding to a fan $\Sigma$ with $n$ rays. Throughout, we use $r$ to denote the rank of a vector bundle on $X(\Sigma)$.

**Theorem 1.1.** Suppose $n > r \geq d$ and $\frac{1}{r} + \frac{1}{n-r} \leq \frac{1}{2}$. Then there is an irreducible toric vector bundle $\mathcal{F}$ of rank $r$ on $X(\Sigma)$ such that the Cox ring of the projectivization $\mathbb{P}(\mathcal{F})$ is not finitely generated.

In particular, on any smooth projective toric surface corresponding to a fan with at least nine rays, there is a rank three toric vector bundle whose projectivization is not a Mori dream space. The bundles that we construct in the proof of Theorem 1.1 are of a special form; in Klyachko’s classification, they correspond to collections of filtrations in which each filtration contains at most one nontrivial subspace, which is required to be of codimension one. The result holds when there are sufficiently many nontrivial subspaces in very general position, and it is sharp in the sense that if $\frac{1}{r} + \frac{1}{n-r} > \frac{1}{2}$ and the nontrivial subspaces are in general position, then the projectivization of any bundle of this form is a Mori dream space. See Corollary 3.4.
Remark 1.2. If $\mathbf{P}(\mathcal{F})$ is a projectivized bundle whose Cox ring is not finitely generated, it may still happen that the section ring of $\mathcal{O}(1)$ on $\mathbf{P}(F)$ is finitely generated. However, Theorem 1.1 implies that there also exist toric vector bundles $\mathcal{F}'$ such that the section ring of $\mathcal{O}(1)$ on $\mathbf{P}(\mathcal{F}')$ is not finitely generated.

Suppose $\mathbf{P}(\mathcal{F})$ is a projectivized toric vector bundle on $X(\Sigma)$ whose Cox ring is not finitely generated, and let $L_1, \ldots, L_k$ be line bundles that positively generate the Picard group of $X(\Sigma)$. Then the section ring of $\mathcal{O}(1)$ on the projectivization of $\mathcal{F}' = F \oplus L_1 \oplus \cdots \oplus L_k$ is not finitely generated. So Theorem 1.1 gives negative answers to Questions 7.1 and 7.2 of [HMP10].

A necessary, but not sufficient, condition for a projective variety to be a Mori dream space is that its pseudoeffective cone be polyhedral. In many of the examples covered by Theorem 1.1, it is unclear whether this condition holds. However, by choosing the toric variety carefully, with an even larger number of rays, we produce examples of projectivized toric vector bundles where this condition fails.

**Theorem 1.3.** Suppose $n - d > r \geq d$ and $\frac{1}{r} + \frac{1}{n-d-r} \leq \frac{1}{2}$, and assume there is some cone $\sigma \in \Sigma$ such that every ray of $\Sigma$ is contained in either $\sigma$ or $-\sigma$. Then there is an irreducible toric vector bundle $\mathcal{F}$ of rank $r$ on $X(\Sigma)$ such that the pseudoeffective cone of $\mathbf{P}(\mathcal{F})$ is not polyhedral.

One can construct examples of toric varieties satisfying the hypotheses of Theorem 1.3 through sequences of iterated blowups of $(\mathbf{P}^1)^d$, as in Example 1.6, below.

The constructions used to prove Theorems 1.1 and 1.3 involve choosing bundles that are very general in their moduli spaces. However, by choosing the fan sufficiently carefully, one gets examples of smooth projective toric varieties in characteristic zero whose projectivized cotangent bundles are not Mori dream spaces. For these examples, the bundle is determined by the combinatorial data in the fan.

**Theorem 1.4.** Suppose $d \geq 3$ and the characteristic of $k$ is not two or three. Then there exists a smooth projective toric variety $X(\Sigma')$ of dimension $d$ over $k$ such that the Cox ring of the projectivized cotangent bundle on $X(\Sigma')$ is not finitely generated.

In this respect, cotangent bundles behave quite differently from tangent bundles, since the Cox ring of the projectivization of the tangent bundle on any smooth toric variety is finitely generated [HS10, Theorem 5.8]. So, Theorem 1.4 shows that there are toric vector bundles $\mathcal{F}$ such that $\mathbf{P}(\mathcal{F})$ is a Mori dream space, but the projectivized dual bundle $\mathbf{P}(\mathcal{F}^\vee)$ is not.

**Remark 1.5.** Throughout, we work over an uncountable field, in order to choose points in very general position. However, examples constructed by Totaro in his work on Hilbert’s 14th Problem over finite fields [Tot08] show that this restriction on the field is not necessary in some cases. For instance, over a field $K$ equal to $\mathbb{Q}$ or $\mathbb{F}_p$ for $p > 3$, there is a set $S$ of nine distinct points in $\mathbb{P}^2(K)$ such that $\text{Bl}_S \mathbb{P}^2$ is not finitely generated. It follows that Theorem 1.4 holds over any extension of these fields. Furthermore, such a set can be chosen in linearly general position, if $K$ is $\mathbb{Q}$ or $\mathbb{F}_p$ for $p > 23$, and it follows that Theorem 1.1 holds over these fields and their extensions in the special case where $r$ is three. Similarly, if $K$ is $\mathbb{Q}$ or $\mathbb{F}_p$ for $p > 7$, there is a set $S'$ of eight points in linearly general position in $\mathbb{P}^3(K)$ such that the Cox ring of $\text{Bl}_{S'} \mathbb{P}^3$ is not finitely generated, and Theorem 1.1 holds over these fields and their extensions in the special case where $r$ is four.
We conclude the introduction with an example of a projectivized rank three bundle on an iterated blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at seven points whose effective cone agrees with the effective cone of $\mathbb{P}^2$ blown up at nine very general points, and hence is not polyhedral.

**Example 1.6.** Let $X(\Sigma)$ be the toric variety obtained by first blowing up one of the toric fixed points on $\mathbb{P}^1 \times \mathbb{P}^1$, then blowing up both of the toric fixed points in the exceptional divisor, and then blowing up all four of the torus fixed points in the new exceptional divisors. The corresponding fan is as shown.

![Fan Diagram](image)

Note that every ray of the fan is contained in either the cone $\sigma$ spanned by $\rho_{10}$ and $\rho_{11}$, or in $-\sigma$, and $\frac{1}{3} + \frac{1}{11-2-3} = \frac{1}{2}$. So $X(\Sigma)$ satisfies the hypotheses of Theorems 1.1 and 1.3.

Let $F$ be a three dimensional vector space, and define filtrations

$$F^p_{\rho_{10}}(j) = \begin{cases} F & \text{for } j \leq 0, \\
F_1 & \text{for } j = 1, \\
0 & \text{for } j > 1, \end{cases}$$

where $F_1, \ldots, F_9$ are two dimension subspaces in very general position, and $F_{10}$ and $F_{11}$ are zero. The subspaces $F_1, \ldots, F_9$ correspond to a set $S = \{p_1, \ldots, p_9\}$ of nine points in very general position in the projective plane $\mathbb{P}_F$ of one dimensional quotients of $F$. Our first main construction, in Section 3, shows that the Cox ring of $\mathbb{P}(\mathcal{O}_F)$ is canonically isomorphic to a polynomial ring in two variables over the Cox ring of the blowup $\text{Bl}_S \mathbb{P}_F$ of the plane at this set of points. Furthermore, in Section 4 we give an isomorphism of class groups $\text{Cl}(\mathbb{P}(\mathcal{O}_F)) \xrightarrow{\sim} \text{Cl}(\text{Bl}_S \mathbb{P}_F)$ that takes $\mathcal{O}(1)$ to the pullback of the hyperplane class of $\mathbb{P}_F$, and the class of $\mathbb{P}(\mathcal{O}_F|_{D_{p_i}})$ to the class of the exceptional divisor $E_{p_i}$ for $i = 1, \ldots, 9$, and show that it induces an identification of the effective cones of the two spaces. Therefore, the pseudoeffective cone of $\mathbb{P}(\mathcal{O}_F)$, like the pseudoeffective cone of $\text{Bl}_S \mathbb{P}_F$, is not polyhedral, and $\mathbb{P}(\mathcal{O}_F)$ is not a Mori dream space.

2. **Preliminaries**

We work over an uncountable field of arbitrary characteristic with the exception of the proof of Theorem 1.4, where we restrict to characteristic not two or three.
Let $T$ be a torus of dimension $d$, with character lattice $M$. Let $X(\Sigma)$ be a smooth projective toric variety with dense torus $T$, and let $\rho_1, \ldots, \rho_n$ be the rays of $\Sigma$. We write $v_j$ for the primitive generator in $N = \text{Hom}(M, \mathbb{Z})$ of the ray $\rho_j$, and $D_{\rho_j}$ for the corresponding prime $T$-invariant divisor in $X(\Sigma)$.

Suppose $F$ is a toric vector bundle of rank $r$ on $X(\Sigma)$. The Klyachko filtrations associated to $F$ are decreasing filtrations of the fiber $F$ over the identity $1_T$, indexed by the rays of $\Sigma$,

$$
\cdots \supset F^{\rho_j}(k-1) \supset F^{\rho_j}(k) \supset F^{\rho_j}(k+1) \supset \cdots ,
$$

and characterized by the following property. If $U_\sigma \subset X(\Sigma)$ is the torus-invariant affine open subvariety corresponding to a cone $\sigma$ in $\Sigma$, and $u$ is a character of the torus, then the space of isotypical sections

$$
H^0(U_\sigma, F)_u = \{ s \in H^0(U_\sigma, F) \mid ts = \chi^u(t)s \text{ for all } t \in T \}
$$

injects into $F$, by evaluation at $1_T$, and the image is

$$
F^\sigma_u = \bigcap_{\rho_j \prec \sigma} F^{\rho_j}((u, v_j)).
$$

These filtrations satisfy the following compatibility condition.

**Klyachko’s Compatibility Condition.** For each maximal cone $\sigma \in \Sigma$, there are characters $u_1, \ldots, u_r \in M$ and a decomposition $F = F_1 \oplus \cdots \oplus F_r$ such that

$$
F^{\rho_j}(k) = \bigoplus_{(u, v_j) \geq k} F_i,
$$

for each $\rho_j \prec \sigma$ and all $k \in \mathbb{Z}$.

The bundle $F$ can be recovered from the family of filtrations $\{F^{\rho_j}(k)\}$, and the induced correspondence between toric vector bundles and finite dimensional vector spaces with compatible families of filtrations is an equivalence of categories. See the original paper [Kly90] or the summary in [Pay08, Section 2] for details.

We write $P(F)$ for the projective bundle $\text{Proj}(\text{Sym}(F))$ of rank one quotients of $F$, and

$$
\pi : P(F) \to X(\Sigma)
$$

for its structure map. The fiber of $P(F)$ over $1_T$ is the projective space $P_F$ of one dimensional quotients of $F$. Following the usual convention, we write $\mathcal{O}(1)$ for the dual of the tautological quotient bundle on $P(F)$, which is relatively ample with respect to $\pi$, and $\mathcal{O}(m)$ for its $m$th tensor power.

Throughout this paper, we will focus on bundles whose filtrations are especially simple, and in particular those of the following form

$$
(*)
$$

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F^{\rho_j}(k) = \begin{cases} 
F & \text{for } k \leq 0, \\
F_j & \text{for } k = 1, \\
0 & \text{for } k > 1,
\end{cases}
$$

where $F_j$ is either 0 or a codimension one subspace of $F$, and all of the nonzero $F_j$ are distinct.

**Lemma 2.1.** Assume $n \geq r$. Let $\{F^{\rho_j}(k)\}$ be a family of filtrations satisfying $(*)$. Then these filtrations satisfy Klyachko’s compatibility condition if and only if, for each maximal cone $\sigma$, the nonzero hyperplanes $F_j$ such that $\rho_j \prec \sigma$ meet transversely in $F$. 

Let $T$ be a torus of dimension $d$, with character lattice $M$. Let $X(\Sigma)$ be a smooth projective toric variety with dense torus $T$, and let $\rho_1, \ldots, \rho_n$ be the rays of $\Sigma$. We write $v_j$ for the primitive generator in $N = \text{Hom}(M, \mathbb{Z})$ of the ray $\rho_j$, and $D_{\rho_j}$ for the corresponding prime $T$-invariant divisor in $X(\Sigma)$.

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**Lemma 2.1.** Assume $n \geq r$. Let $\{F^{\rho_j}(k)\}$ be a family of filtrations satisfying $(*)$. Then these filtrations satisfy Klyachko’s compatibility condition if and only if, for each maximal cone $\sigma$, the nonzero hyperplanes $F_j$ such that $\rho_j \prec \sigma$ meet transversely in $F$. 

Proof. Let $\sigma$ be a maximal cone in $\Sigma$. If the filtrations satisfy Klyachko's compatibility condition, then there is a splitting $F = G_1 \oplus \cdots \oplus G_r$ such that each nonzero $F_j$ is a sum of $r - 1$ of the $G_i$, for $\rho_j \prec \sigma$. Furthermore, the nonzero $F_j$ are distinct by hypothesis so, after renumbering, we may assume $F_j = G_1 \oplus \cdots \oplus \hat{G}_j \oplus \cdots \oplus G_r$. Then the nonzero $F_j$ such that $\rho_j \prec \sigma$ meet transversely along the sum of those $G_i$ such that $F_i$ is zero or $\rho_i$ is not a ray of $\sigma$.

For the converse, suppose the nonzero $F_j$ corresponding to rays $\rho_j \prec \sigma$ meet transversely. Then we can choose coordinates on $F$ so that each such $F_j$ is a coordinate hyperplane. After renumbering we may assume the one dimensional coordinate subspaces $L_1, \ldots, L_r$ are such that $F_j = L_1 \oplus \cdots \oplus \hat{L}_j \oplus \cdots \oplus L_r$, for each nonzero $F_j$ such that $\rho_j \prec \sigma$. Since $X(\Sigma)$ is smooth, we can choose $u_i \in M$, so that for each $\rho_j \prec \sigma$, the product $(u_i, v_j)$ is one if $F_j$ is nonzero and $i \neq j$, and is zero otherwise. Then the characters $u_1, \ldots, u_r$ and the decomposition $F = L_1 \oplus \cdots \oplus L_r$ satisfy Klyachko's compatibility condition.

Since at most $r$ hyperplanes can meet transversely in a vector space of rank $r$, the condition that $r \geq d$ is necessary for a collection of filtrations satisfying (*) to define a toric vector bundle on a smooth projective toric variety, if all of the $F_j$ are nonzero. If the $F_j$ are chosen in general position, then the condition $r \geq d$ is also sufficient.

One reason for working with bundles given by filtrations of this form is that the space of $T$-invariant global sections of $O(m)$, and the orders of vanishing of these sections along the divisors $\pi^{-1}(D_{\rho_i})$ are particularly easy to understand. See Lemmas 4.2 and 4.3.

3. Torus quotients and Cox rings

Let $X$ be a smooth variety whose divisor class group is finitely generated and torsion free. Choose divisors $D_1, \ldots, D_k$ whose classes form a basis for the class group $\text{Cl}(X)$. Then the Cox total coordinate ring of $X$ is

$$\mathcal{R}(X) = \bigoplus_{(m_1, \ldots, m_k) \in \mathbb{Z}^k} H^0(X, O(m_1 D_1 + \cdots + m_k D_k)),$$

with the natural multiplication map given by tensor products of global sections. See [HK00] for further details and a discussion of the special properties of Mori dream spaces, those varieties whose Cox rings are finitely generated. If $X_0 \subset X$ is an open subvariety whose complement has codimension at least two, then $\text{Cl}(X_0)$ and $\mathcal{R}(X_0)$ are naturally identified with $\text{Cl}(X)$ and $\mathcal{R}(X)$, respectively.

Remark 3.1. Cox rings can be defined in greater generality, for possibly singular and nonseparated prevartieties whose class groups are finitely generated, but may contain torsion [Hau08]. Smooth and separated varieties with torsion free class groups suffice for all of the purposes of this paper.

Our main tool for understanding Cox rings of projectivized toric vector bundles will be the following presentation of Cox rings for certain varieties with torus actions.
Proposition 3.2. Let $X$ be a smooth variety with a $T$-action. Suppose $D_1, \ldots, D_h$ are irreducible divisors in $X$ with positive dimensional generic stabilizers. Suppose, furthermore, that $T$ acts freely on $X \setminus (D_1 \cup \cdots \cup D_h)$ with geometric quotient a smooth variety $Y$, and that the class group of $Y$ is torsion free. Then $R(X)$ is isomorphic to a polynomial ring in $s$ variables over $R(Y)$.

Proof. This is the special case of [HS10, Theorem 1.1] where $X$ is smooth, the $T$-action on the complement of $D_1 \cup \cdots \cup D_h$ is free, and the geometric quotient $Y$ is separated, with torsion free class group. □

Let $F$ be a toric vector bundle on $X$ given by filtrations satisfying the condition (*) discussed in Section 2. After renumbering, say the $F_i$ are distinct hyperplanes for $i \leq s$, and $F_j$ is zero for $j > s$, and let $S = \{p_1, \ldots, p_s\}$ be the set of points corresponding to $F_1, \ldots, F_s$ in the projective space $\mathbb{P}_F$.

Theorem 3.3. The Cox ring $R(\mathbb{P}(F))$ is isomorphic to a polynomial ring in $n - s$ variables over $R(\text{Bl}_S \mathbb{P}_F)$.

Proof. For $j > s$, the divisor $D_j = \pi^{-1}(D_{p_j})$ in $\mathbb{P}(F)$ is stabilized pointwise by the one parameter subgroup corresponding to $\nu_j$. To prove the theorem, we produce closed subsets $Z$ and $W$ of codimension two in $\mathbb{P}(F)$ and $\text{Bl}_S \mathbb{P}_F$, respectively, such that $T$ acts freely on the complement of $Z \cup D_{s+1} \cup \cdots \cup D_n$, with geometric quotient $\text{Bl}_S \mathbb{P}_F \setminus W$. The isomorphism of Cox rings then follows from Proposition 3.2, since the class group of $\text{Bl}_S \mathbb{P}_F$ is torsion free and the Cox rings of $\mathbb{P}(F)$ and $\text{Bl}_S \mathbb{P}_F$ are canonically identified with those of $\mathbb{P}(F) \setminus Z$ and $\text{Bl}_S \mathbb{P}_F \setminus W$, respectively.

We determine a collection of codimension two subvarieties in $\mathbb{P}(F)$ and $\text{Bl}_S \mathbb{P}_F$, as follows. For each $1 \leq i \leq s$, choose coordinates on $F$ so that $F_i$ is a coordinate subspace. The two-dimensional coordinate subspaces determine an arrangement $Z_i$ of codimension two linear subspaces in the fiber $\mathbb{P}_F$ over $1_T$, containing $p_i$. Also, the one dimensional coordinate subspaces contained in $F_i$ determine an arrangement $W_i$ of hyperplanes in the exceptional divisor $E_i \cong \mathbb{P}_{F_i}$ over $p_i$. Similarly, the choice of coordinates determines a splitting of the restriction of $F$ to the dense $T$-orbit $O_{p_i}$ in $D_{p_i}$, and the one dimensional coordinate subspaces give an arrangement $W_i$ of codimension one subbundles in $\mathbb{P}(F|_{O_{p_i}})$. Finally, let $V$ be the union of the codimension two $T$-invariant subvarieties in $X(\Sigma)$. We then set

$$Z = \pi^{-1}(V) \cup (\overline{W_1} \cup \cdots \cup \overline{W_s}) \cup (T\overline{Z_1} \cup \cdots \cup T\overline{Z_s}).$$

We denote again by $Z_i$ the strict tranforms of $Z_i$ in the all the blow ups $\text{Bl}_{p_i} \mathbb{P}(F)$ and $\text{Bl}_S \mathbb{P}(F)$, and we also set

$$W = (W_1 \cup \cdots \cup W_s) \cup (Z_1 \cup \cdots Z_s) \subseteq \text{Bl}_S \mathbb{P}(F).$$

We now claim that $T$ acts freely on the open subset

$$X_0 = \mathbb{P}(F) \setminus (Z \cup D_{s+1} \cup \cdots \cup D_n),$$

with geometric quotient $\text{Bl}_S \mathbb{P}_F \setminus W$. The proof of this claim is through a local toric computation.

The choice of coordinates above, for each $i$, induces an affine toric structure on the restriction of $F$ to the open subset $U_i$ corresponding to $p_i$, and a toric structure of $\mathbb{P}(F|_{U_i})$, which can be described as follows. See [Oda88, pp. 58–59].
Number the coordinates so that $F_i$ is given by the vanishing of the last coordinate. Then the affine toric structure on $F_i|U_i$ corresponds to the cone $\sigma_i$ in $N_\mathbb{R} \times \mathbb{R}^r$ whose rays are spanned by $(0, e_1), \ldots, (0, e_r), \text{ and } (v_i, e_1 + \ldots + e_{r-1})$. The projectivization $P(F_i|U_i)$ then corresponds to the fan in $N_\mathbb{R} \times (\mathbb{R}^r/(1, \ldots, 1))$ whose maximal cones are the projections of the facets of $\sigma_i$ containing $(v_i, e_1 + \ldots + e_{r-1})$. Now the closed subset $W_i \cup TZ_i$ is exactly the union of the codimension two torus invariant subvarieties of $P(F_i|U_i)$, with this toric structure, so the complement is the toric variety corresponding to the fan $\Sigma_i$ consisting of the projections of the rays of $\sigma_i$.

Projecting to $\mathbb{R}^r/(1, \ldots, 1)$ maps distinct rays of $\Sigma_i$ to distinct rays, and hence the corresponding morphism of toric varieties is a geometric quotient for the action of the torus $T$, which corresponds to the kernel of this linear projection [ANH99, Proposition 3.2]. The image of this projection is the set of rays in the fan corresponding to the blowup of the projective space $\mathbb{P}^r$ at the point $p_i$ corresponding to the coordinate hyperplane $F_i$, and the image of the corresponding map of toric varieties
\[
\phi_i : (P(F_i|U_i)) \setminus (W_i \cup TZ_i) \rightarrow Bl_{p_i} P_F
\]
is exactly $Bl_{s_i} P_F \setminus (W_i \cup Z_i)$, the complement of the codimension two torus invariant subvarieties for this choice of coordinates. Furthermore, for the rest of $Z$ and $W$, the preimage under $\phi_i$ composed with the projection to $P_F$ of $(W_1 \cup \cdots \cup \tilde{W}_i \cup \cdots \cup W_s) \cup (Z_1 \cup \cdots \cup \tilde{Z}_i \cup \cdots \cup Z_s)$ is exactly $(\tilde{W}_i \cup \cdots \cup \tilde{W}_i \cup \cdots \cup \tilde{W}_i) \cup (TZ_1 \cup \cdots \cup TZ_i \cup \cdots \cup TZ_s)$, so $\phi_i$ restricts to a geometric quotient map from an open subset of $P(F) \setminus Z$ to an open subset of $Bl_S P_F$. The union of the bundles $P(F_i|U_i)$ is exactly $P(F) \setminus \pi^{-1}(V)$ and, since the maps $\phi_i$ are geometric quotients, they canonically glue on overlaps to give a morphism
\[
\phi : P(F) \setminus Z \rightarrow Bl_S P_F \setminus W.
\]
Now $Bl_S P_F \setminus W$ is covered by the open sets $B_i = Bl_{s_i} P_F \setminus (W \cup E_1 \cup \cdots \cup \tilde{E}_i \cup \cdots \cup E_s)$. Since the $F_i$ are distinct, the preimage of $B_i$ is contained in $P(F_i|U_i)$. Therefore, the restriction of $\phi$ to the preimage of $B_i$ is a geometric quotient, induced by restricting $\phi_i$. Since the property of being a geometric quotient is local on the base, this proves the claim, and hence the theorem.

**Proof of Theorem 1.1.** We can take $\mathcal{F}$ as in Theorem 3.3 with $\mathcal{R}(Bl_S P_F)$ not finitely generated [Muk04], and then the conclusion follows from that theorem. □

**Corollary 3.4.** Suppose $\mathcal{F}$ is given by filtrations satisfying (*) with the hyperplanes $F_i$ in general position. If $\frac{1}{r} + \frac{1}{n_1-r} > \frac{1}{2}$ then $P(\mathcal{F})$ is a Mori dream space.

**Proof.** Suppose $\frac{1}{r} + \frac{1}{n_1-r} > \frac{1}{2}$. Then the blow up of $P^{r-1}$ at $n$ points in general position is a Mori dream space [CT06, Theorem 1.3], and then so is the blow up $Bl_{s_i} P^{r-1}$ of $P^{r-1}$ at $s$ points in general position, where $s$ is the number of rays $\rho_j$ such that $F_j$ is nonzero. The corollary then follows immediately from Theorem 3.3, which says that $\mathcal{R}(P(\mathcal{F}))$ is finitely generated over $\mathcal{R}(Bl_{s_i} P^{r-1})$. □

If the points $p_1, \ldots, p_s$ are not in general position then $P(\mathcal{F})$ can be a Mori dream space, even when $\frac{1}{r} + \frac{1}{n_1-r} \leq \frac{1}{2}$. For instance, if $p_1, \ldots, p_s$ are collinear then $Bl_S P_F$ is a rational variety with a torus action with orbits of codimension one, and hence is a Mori dream space [HS10, Ott10]. Also, if $p_1, \ldots, p_s$ lie on a rational normal curve, then $Bl_S P_F$ is a Mori dream space [CT06, Theorem 1.2].
We now give an example of a smooth projective toric threefold whose projectivized cotangent bundle is not a Mori dream space. The construction uses a particularly nice configuration of nine points in $\mathbb{Z}^3$, due to Totaro, such that, for any field $k$ of characteristic not two or three, the blowup of $\mathbb{P}^2(k)$ at the corresponding nine $k$-points is not a Mori dream space. The proof of Theorem 1.4 will be by induction on dimension, starting from this example.

Example 3.5. In this example, we work over a field $k$ of characteristic not two or three.

The vectors
\[ v_1 = (0,0,1), \quad v_2 = (0,1,0), \quad v_3 = (1,1,1), \quad v_4 = (-1,-2,-2) \]
span the four rays of a unique complete fan $\Sigma_4$ in $\mathbb{R}^3$. The corresponding toric variety $X(\Sigma_4)$ is isomorphic to $\mathbb{P}^3$. Consider the vectors
\[ v_5 = (1,1,2), \quad v_6 = (0,-1,1), \quad v_7 = (1,0,1), \quad v_8 = (1,-1,1), \]
\[ v_9 = (-1,-2,-1), \quad v_{10} = (-1,-1,0), \quad v_{11} = (-1,-1,1), \quad v_{12} = (-1,0,1), \]
\[ v_{13} = (-1,1,1), \quad v_{14} = (0,1,1), \]
and let $\Sigma_i$ be the stellar subdivision of $\Sigma_{i-1}$ along the ray spanned by $v_i$, for $5 \leq i \leq 14$. For each such $i$, the vector $v_i$ is the sum of two or three of the $v_j$ that span a cone in $\Sigma_{i-1}$. Therefore, the toric variety $X(\Sigma_i)$ is the blowup of $X(\Sigma_{i-1})$ at either a point or a torus invariant smooth rational curve. In particular, if we set $\Sigma = \Sigma_{14}$, then the corresponding toric variety $X(\Sigma)$ is smooth and projective. The twist $\mathcal{F}$ of the cotangent bundle on $X(\Sigma)$ by the anticanonical bundle $\mathcal{O}(D_{\rho_1} + \cdots + D_{\rho_{14}})$ is given by the vector space $F = k^3$ with filtrations
\[ F^\rho(j) = \begin{cases} 
  k^3 & \text{for } j \leq 0, \\
  v_i^j & \text{for } j = 1, \\
  0 & \text{for } j > 1, 
\end{cases} \]
Since the characteristic of $k$ is not two or three, the vectors $v_i^j$ are all distinct in $\mathbb{P}^2_k$, and hence the filtrations satisfy (*). Twisting by a line bundle does not change the projectivization, so Proposition 3.2 says that the Cox ring of the projectivized cotangent bundle of $X(\Sigma)$ is isomorphic to the Cox ring of $\text{Bl}_S \mathbb{P}^2_k$, where $S = \{v_1^+, \ldots, v_{14}^+\}$. The subset
\[ S' = \{v_1^+, v_3^+, v_5^+, v_7^+, v_8^+, v_{11}^+, v_{12}^+, v_{13}^+, v_{14}^+\} \]
is the complete intersection of two smooth cubics, and the Cox ring of $\text{Bl}_{S'} \mathbb{P}^2_k$ is not finitely generated [Tot08, Theorem 2.1 and Corollary 5.1]. It follows that $\text{Bl}_S \mathbb{P}^2_k$ is not a Mori dream space, and neither is the projectivized cotangent bundle of $X(\Sigma)$.

We use the following lemma on Cox rings of blowups of projective space at finitely many points in a hyperplane in the proof of Theorem 1.4. Instances of this basic fact have appeared, including in [HIT04, Example 1.8]. However, lacking a suitable reference, we give a proof.

Lemma 3.6. Let $S$ be a finite set of points contained in a hyperplane $H$ in $\mathbb{P}^d$. Then the Cox ring of $\text{Bl}_S \mathbb{P}^d$ is isomorphic to a polynomial ring in one variable over the Cox ring of $\text{Bl}_S H$.

Proof. Choose coordinates on $\mathbb{P}^d$ so that $H$ is a coordinate hyperplane, and let $G_m$ act by scaling on the coordinate that cuts out $H$. The action of $G_m$ lifts to an action on $\text{Bl}_S \mathbb{P}^d$, and we let $Y$ be the locus of fixed points of this action. Then $G_m$
acts freely on \( \text{Bl}_S \mathbf{P}^d \setminus Y \), with quotient \( \text{Bl}_S H \). The strict transform of \( H \) is the only divisor contained in \( Y \), so the lemma follows by applying Proposition 3.2. \( \square \)

Proof of Theorem 1.4. Let \( k \) be a field of characteristic not two or three. We must show that, for each dimension \( d \geq 3 \), there is a fan \( \Sigma \) in \( \mathbb{R}^d \) such that

1. The toric variety \( X(\Sigma) \) is smooth and projective.
2. The hyperplanes in \( k^d \) perpendicular to the primitive generators of the rays of \( \Sigma \) are distinct.
3. The Cox ring of \( \text{Bl}_S \mathbf{P}^d_k \) is not finitely generated, where \( S \) is the set of points corresponding to these hyperplanes.

For \( d = 3 \), we have Example 3.5, and we proceed by induction.

Suppose \( \Sigma \) is a fan in \( \mathbb{R}^d \) satisfying (1), (2), and (3). Embed \( \mathbb{R}^d \) as the last coordinate hyperplane in \( \mathbb{R}^{d+1} \), and let \( \Sigma' \) be the fan in \( \mathbb{R}^{d+1} \) whose maximal cones are spanned by a maximal cone of \( \Sigma \) together with either \((1,...,1)\) or \((1,...,1,-1)\). The corresponding toric variety \( X(\Sigma') \) is smooth and projective and, since the characteristic of \( k \) is not two, the hyperplanes in \( k^{d+1} \) perpendicular to the rays of \( \Sigma' \) are distinct. It remains to show that \( \Sigma' \) satisfies (3). Let \( S' \) be the corresponding set of points in \( \mathbb{P}^d_k \). Now \( S' \) contains the subset \( S \) of points corresponding to rays of \( \Sigma \), and \( S \) is contained in a hyperplane \( H \). By hypothesis, the Cox ring of \( \text{Bl}_S \mathbf{P}^d_k \) is not finitely generated. By Lemma 3.6, it follows that \( \text{Bl}_S \mathbf{P}^d_k \) is not a Mori dream space, and neither is \( \text{Bl}_{S'} \mathbf{P}^d_k \). The theorem follows, since the Cox ring of the projectivized cotangent bundle of \( X(\Sigma) \) is isomorphic to the Cox ring of \( \text{Bl}_{S'} \mathbf{P}^d_k \), by Proposition 3.2. \( \square \)

4. Pseudoeffective cones

The pseudoeffective cone of a projective variety \( X \) is the closure of the cone spanned by the classes of all effective divisors in the space of numerical equivalence classes of divisors \( N^1(X)_\mathbb{R} = N^1(X) \otimes \mathbb{Z} \mathbb{R} \). For projectivized toric vector bundles and for blow ups of projective spaces at finite sets of points, linear equivalence and numerical equivalence coincide and then we identify \( N^1(X)_\mathbb{R} \) and \( \text{Cl}(X)_\mathbb{R} = \text{Cl}(X) \otimes \mathbb{Z} \mathbb{R} \).

Lemma 4.1. Every effective divisor on \( \mathbf{P}(\mathcal{F}) \) is linearly equivalent to a torus invariant effective divisor.

Proof. Let \( D \) be an effective divisor in \( \mathbf{P}(\mathcal{F}) \), and let \( \gamma_1, \ldots, \gamma_d \) be a basis for the lattice of one parameter subgroups of \( T \). Set \( D_0 = D \), and let \( D_i \) be the limit as \( t \) goes to zero of \( \gamma_i(t)D_{i-1} \), for \( 1 \leq i \leq d \). Then \( D_i \) is effective, linearly equivalent to \( D \), and invariant under the subgroup of \( T \) generated by \( \gamma_1, \ldots, \gamma_i \). In particular, \( D_d \) is an effective \( T \)-invariant divisor linearly equivalent to \( D \). \( \square \)

So the pseudoeffective cone of \( \mathbf{P}(\mathcal{F}) \) is the closure of the cone generated by classes of torus invariant prime divisors. Note that every torus invariant prime divisor in \( \mathbf{P}(\mathcal{F}) \) is either the preimage of a torus invariant prime divisor in \( X(\Sigma) \) or surjects onto \( X(\Sigma) \). If a torus invariant prime divisor surjects onto \( X(\Sigma) \) then it must be the closure of the torus orbit of its intersection with the fiber over the identity. We write \( \mathcal{D}_H \) for the closure of the torus orbit of a hypersurface \( H \) in \( \mathbf{P}(\mathcal{F}) \).

One key step toward understanding the pseudoeffective cone of \( \mathbf{P}(\mathcal{F}) \) is to express the class of each such \( \mathcal{D}_H \) as a linear combination of \( \mathcal{O}(1) \) and the \( \pi^{-1}(D_{\rho_i}) \). Such
expressions may be somewhat complicated in general, but are relatively simple for bundles given by filtrations of the special form discussed in Section 2.

Suppose the filtrations \{F^\rho_0(j)\} satisfy (*).

**Lemma 4.2.** Restriction to the fiber \(P_F\) gives an isomorphism from the space of \(T\)-invariant global sections of \(O(m)\) on \(P(F)\) to \(\text{Sym}^m(F)\).

**Proof.** For any bundle \(F\), global sections of \(O(m)\) on \(P(F)\) are naturally identified with global sections of \(\text{Sym}^m F\). Now, \(\text{Sym}^m F\) is a toric vector bundle, with fiber \(\text{Sym}^m F\) over \(1_F\), and since the filtrations defining \(F\) satisfy (*), the filtrations defining \(\text{Sym}^m F\) are given by

\[
\text{Sym}^m F^\rho_0(j) = \begin{cases} 
\text{Sym}^m F & \text{for } j \leq 0, \\
\text{Image (Sym}^i F_i \otimes \text{Sym}^{m-i} F \rightarrow \text{Sym}^m F) & \text{for } 1 \leq j \leq m, \\
0 & \text{for } j > m.
\end{cases}
\]

The space of \(T\)-invariant sections of \(\text{Sym}^m F\) is the intersection of all of these filtrations evaluated at zero, and the lemma follows, because \(\text{Sym}^m F^\rho_0(0)\) is \(\text{Sym}^m F\), for every ray \(\rho_i\). \(\square\)

Let \(p_i\) be the point in \(P_F\) corresponding to the one dimensional quotient \(p_i = F/F_i\), whenever \(F_i\) is nonzero. As in the proof of Theorem 3.3, we write \(D_j\) for the \(T\)-invariant prime divisor \(\pi^{-1}(D_{\rho_j})\) in \(P(F)\).

**Lemma 4.3.** Let \(H\) be a hypersurface of degree \(m\) in \(P_F\), and let \(m_i\) be the multiplicity of \(H\) at \(p_i\). Then there is a linear equivalence

\[
D_H \sim O(m) - \sum_i m_i(\pi^{-1}(D_{\rho_i})),
\]

where the sum is over those \(i\) such that \(F_i\) is nonzero.

**Proof.** Let \(h \in \text{Sym}^m F\) be a defining equation for \(H\). Then \(h\) corresponds to a torus invariant section \(s\) of \(O(m)\) on \(P(F)\), by Lemma 4.2. If \(F_i\) is zero then \(s\) does not vanish along \(D_i\) and if \(F_i\) is zero then \(m_i\) is the largest integer such that \(h\) is contained in the image of \(\text{Sym}^{m_i} F_i \otimes \text{Sym}^{m-m_i} F\) in \(\text{Sym}^m F\). The one parameter subgroup corresponding to \(v_i\) extends to an embedding of the affine line \(A^1\) in \(X(\Sigma)\) meeting \(D_{\rho_i}\) transversely at the image of zero. After restricting the section \(s\) to the preimage of \(A^1\), we must show that its order of vanishing along the preimage of zero is \(m_i\). The isotypical decomposition of the module of global sections of \(O(1)\) on the preimage of \(A^1\), for the action of the one-parameter subgroup corresponding to \(v_i\), is exactly \(\bigoplus_j F^\rho_0(j)\), and multiplication by the coordinate \(x\) on \(A^1\) decreases degree by one. The sections of \(O(m)\) are given by the \(m\)th symmetric power of this module, in which the image of \(\text{Sym}^k F_i \otimes \text{Sym}^{m-k} F\) in \(\text{Sym}^m F\) appears in degree \(k\), for nonnegative integers \(k\). It follows that the \(T\)-invariant section \(s\) is equal to \(x^{m_i}\) times a section that is nonvanishing along the preimage of zero, and hence vanishes to order \(m_i\), as required. \(\square\)

Now, we fix a maximal cone \(\sigma\) and, after renumbering, we may assume \(\sigma\) is spanned by \(\rho_1, \ldots, \rho_d\). Furthermore, for the remainder of the paper we assume that

\[
F_i = 0, \text{ for } 1 \leq i \leq d.
\]

The class of \(O(1)\) and the classes of \(D_{d+1}, \ldots, D_n\) form a basis for \(\text{Cl}(P(F))\).
Let \( f : \text{Bl}_S \mathbf{P}_F \to \mathbf{P}_F \) be the blowup of \( \mathbf{P}_F \) at the finite set of distinct points \( \{ p_i \} \), corresponding to the nonzero \( F_i \), for \( i > d \). Let \( L \) be a hyperplane in \( \mathbf{P}_F \), and let \( E_i \) be the exceptional divisor over \( p_i \). Then \( f^*L \) and \( \{ E_i \} \) together form a basis for \( \text{Cl}(\text{Bl}_S \mathbf{P}_F) \).

We consider the linear map \( \varphi : \text{Cl}(\text{Bl}_S \mathbf{P}_F)_\mathbb{R} \to \text{Cl}(\mathbf{P}(\mathcal{F}))_\mathbb{R} \), taking \( f^*L \) to \( \mathcal{O}(1) \) and the class of \( E_i \) to the class of \( D_i \), for \( i > d \). If \( H \) is a hypersurface of degree \( m \) in \( \mathbf{P}_F \) passing through \( p_i \) with multiplicity \( m_i \), then the class of the strict transform of \( H \) in \( \text{Bl}_S \mathbf{P}_F \) is \( f^*mL - \sum_i m_i E_i \). So Lemma 4.3 says that \( \varphi \) maps the class of the strict transform of \( H \) to the class of \( \mathcal{D}_H \).

**Proposition 4.4.** The pseudoeffective cone of \( \mathbf{P}(\mathcal{F}) \) is generated by the image under \( \varphi \) of the pseudoeffective cone of \( \text{Bl}_S \mathbf{P}_F \) together with the classes of those \( D_i \) such that \( F_i \) is zero.

**Proof.** By Lemma 4.1, every effective divisor on \( \mathbf{P}(\mathcal{F}) \) is in the cone generated by the classes \( \mathcal{D}_H \), for hypersurfaces \( H \) in \( \mathbf{P}_F \), and the classes \( D_i \). On \( \text{Bl}_S \mathbf{P}_F \), every effective divisor is in the cone generated by the classes of the strict transforms of the hypersurfaces \( H \) in \( \mathbf{P}_F \), and the classes \( E_i \). Now, the classes \( D_i \) such that \( F_i \) is nonzero are the images under \( \varphi \) of the classes \( E_i \), and Lemma 4.3 says that the class of \( \mathcal{D}_H \) is the image under \( \varphi \) of the strict transform of the hypersurface \( H \) in \( \mathbf{P}_F \). Therefore, the cone of effective classes on \( \mathbf{P}(\mathcal{F}) \) is equal to the cone generated by the image under \( \varphi \) of the cone of effective classes on \( \text{Bl}_S \mathbf{P}_F \) together with the classes of those \( D_i \) such that \( F_i \) is zero. The proposition follows by taking closures.

**Proof of Theorem 1.3.** Let \( \sigma \) be the cone spanned by \( \rho_1, \ldots, \rho_d \), and choose the toric variety \( X(\Sigma) \) so that each of the other rays \( \rho_i \) is contained in \( -\sigma \). This can be accomplished, as in Example 1.6, by taking a suitable sequence of blowups of \( (\mathbf{P}^1)^d \). Choose the filtrations defining \( \mathcal{F} \) so that \( F_{d+1}, \ldots, F_n \) are distinct hyperplanes, and \( F_i = 0 \) for \( i \leq d \).

The choice of the filtrations ensures that \( \varphi \) is an isomorphism on class groups, since it maps the basis elements \( f^*L, E_{d+1}, \ldots, E_n \) for \( \text{Cl}(\text{Bl}_S \mathbf{P}_F) \) to the basis elements \( \mathcal{O}(1), D_{d+1}, \ldots, D_n \) for \( \text{Cl}(\mathbf{P}(\mathcal{F})) \), respectively. Furthermore, the choice of the fan \( \Sigma \) ensures that, for \( i \leq d \), the divisor \( D_{\rho_i} \) is linearly equivalent to an effective combination of the \( D_{\rho_j} \), for \( j > d \). So the classes of \( D_1, \ldots, D_d \) are in the cone spanned the classes of \( D_i \) for \( i > d \), and hence are in the image under \( \varphi \) of the pseudoeffective cone of \( \text{Bl}_S \mathbf{P}_F \). Therefore, by Proposition 4.4, the linear isomorphism \( \varphi \) identifies the pseudoeffective cone of \( \text{Bl}_S \mathbf{P}_F \) with the pseudoeffective cone of \( \mathbf{P}(\mathcal{F}) \). If \( F_{d+1}, \ldots, F_n \) are in very general position, then the inequalities on \( r \) and \( n \) imply that the pseudoeffective cone of \( \text{Bl}_S \mathbf{P}_F \) is not polyhedral [Muk04], and the theorem follows.

**References**


