Appendix to: Generalized Stability of Kronecker Coefficients

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Contents

In this appendix to [**S**], we provide some complementary remarks and observations that didn't make it into the paper. We follow the same terminology and notation. External references to numbered equations etc. are pointers to [**S**].

A. Line reduction

Since I_m is the trivial representation of S_m, one knows that $g(\alpha,\beta,m)=\delta_{\alpha,\beta}$.

By taking V_3 to be one-dimensional and $\gamma=(m)$, we may deduce from (2.1) that

$$S^m(V_1\otimes V_2)\ \cong\ \bigoplus_{\alpha} V_1(\alpha)\otimes V_2(\alpha)\qquad (\text{as } \mathrm{gl}(V_1)\oplus\mathrm{gl}(V_2)\text{-modules}),$$

where α ranges over partitions with $\ell(\alpha)\leqslant\min(\dim V_1,\dim V_2)$. In particular,

$$S^m(V_1\otimes V_2\otimes V_3)\ \cong\ \bigoplus_{\gamma}(V_1\otimes V_2)(\gamma)\otimes V_3(\gamma) \tag{A.1}$$

as $\mathrm{gl}(V_1)\oplus\mathrm{gl}(V_2)\oplus\mathrm{gl}(V_3)$-modules. This yields a well-known reformulation of (2.1).

Proposition A.1. *If $\ell(\alpha)\leqslant\dim V_1$ and $\ell(\beta)\leqslant\dim V_2$, then the Kronecker coefficient $g(\alpha\beta\gamma)$ is the multiplicity of $V_1(\alpha)\otimes V_2(\beta)$ in the $\mathrm{gl}(V_1)\oplus\mathrm{gl}(V_2)$-module $(V_1\otimes V_2)(\gamma)$.*

If V is m-dimensional, then $V(1^m)$ is the one-dimensional $\mathrm{gl}(V)$-module carried by the trace map $\mathrm{gl}(V)\to\mathbb{C}$. It follows easily that $V(n^m)\cong V(1^m)^{\otimes n}$ and more generally

$$V(\mu)\otimes V(1^m)\cong V(\mu+1^m) \tag{A.2}$$

for all partitions μ with at most m parts.

Proposition A.2. *Assume $\ell(\alpha) \leqslant a$ and $\ell(\beta) \leqslant b$.*

(a) *(Well known.) If $g(\alpha\beta\gamma) > 0$, then $\ell(\gamma) \leqslant ab$.*
(b) *(Vallejo [**V1**].) We have $g(\alpha\beta\gamma) = g(\alpha + b^a, \beta + a^b, \gamma + 1^{ab})$.*
(c) *If $\ell(\gamma) = ab$ and $g(\alpha\beta\gamma) > 0$, then α has at least b columns of length a and β has at least a columns of length b.*

Proof. Choose vector spaces V_1 and V_2 of dimensions a and b.

(a) All of the nonzero summands in (A.1) have $\ell(\gamma) \leqslant ab$.

(b) We may assume $\ell(\gamma) \leqslant ab$; otherwise (a) implies that both Kronecker coefficients are zero. One can check that

$$(V_1 \otimes V_2)(1^{ab}) \cong V_1(b^a) \otimes V_2(a^b), \tag{A.3}$$

so the claimed formula follows by repeated application of (A.2).

(c) If $\ell(\gamma) = ab$ then $\gamma = \hat{\gamma} + 1^{ab}$ for some partition $\hat{\gamma}$ with at most ab parts. It follows that all summands $V_1(\alpha) \otimes V_2(\beta)$ in the irreducible decomposition of $(V_1 \otimes V_2)(\gamma)$ are obtained by tensoring the summands of $(V_1 \otimes) V_2(\hat{\gamma})$ by $(V_1 \otimes V_2)(1^{ab})$. The claim now follows via (A.2) and (A.3). □

Thus if the length of one partition in a triple $\alpha\beta\gamma$ matches the product of the lengths of the other two, then the first column of that partition and appropriate numbers of columns of the other two may be deleted without changing the Kronecker coefficient. By conjugation invariance, similar reductions are also possible if (say) $\ell(\gamma) = \alpha_1\beta_1$ or $\gamma_1 = \alpha_1\ell(\beta)$. We call any such operation a *line reduction*.

B. Complementation

If $\ell(\mu) \leqslant m = \dim V$, then the dual $\mathrm{gl}(V)$-module $V(\mu)^*$ is isomorphic to $V(\mu^*)$, where

$$\mu^* := (-\mu_m, \dots, -\mu_2, -\mu_1).$$

Of course μ^* will generally not be a partition, but (A.2) implies

$$V(\mu^*) \otimes V(n^m) \cong V(n^m + \mu^*),$$

and $\nu = n^m + \mu^*$ will be a partition as long as $n \geqslant \mu_1$. Indeed, ν is the partition one obtains by removing the diagram of μ from an $m \times n$ rectangle and rotating the result by 180 degrees. We call ν the n^m*-complement* of μ.

In the following result, the dual weights α^*, β^*, and γ^* should be understood as associated with vector spaces of dimensions a, b, and ab (respectively).

Proposition B.1. *If $\ell(\alpha) \leqslant a$, $\ell(\beta) \leqslant b$, and $\gamma_1 \leqslant n$, then*

$$g(\alpha\beta\gamma) = g((bn)^a + \alpha^*, (an)^b + \beta^*, n^{ab} + \gamma^*).$$

Furthermore, $\alpha \subseteq (bn)^a$, $\beta \subseteq (an)^b$, and $\gamma \subseteq n^{ab}$, or else $g(\alpha\beta\gamma) = 0$.

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	0	1	0	1	0	1	0	1	0	1
3	0	1	1	2	1	3	2	4	3	5	4	7
4	0	1	1	5	4	16	21	67	118	307	630	1495
5	0	0	1	6	21	216	1890	18371	167596	1437657		
6	0	0	1	13	158	9309	445442	20969042				
7	0	0	0	14	1456	438744	125250433					
8	0	0	1	18	9854	17957625						
9	0	0	1	14	44852							

TABLE C.1: The Kronecker coefficients $G(m,n) = g(n^m, n^m, n^m)$.

Proof. To see the necessity of the rectangle-fitting, note that if γ failed to fit in the rectangle n^{ab}, then $g(\alpha\beta\gamma) = 0$ by Proposition A.2. If (say) α failed to fit in the rectangle $(bn)^a$, then α' would have length greater than bn, and hence $g(\alpha\beta\gamma) = g(\beta\gamma'\alpha') = 0$ by conjugation invariance and Proposition A.2.

Now choose vector spaces V_1 and V_2 of dimensions a and b. If we dualize Proposition A.1, we see that $g(\alpha\beta\gamma)$ is the multiplicity of $V_1(\alpha^*) \otimes V_2(\beta^*)$ in $(V_1 \otimes V_2)(\gamma^*)$. If we take the tensor product of the latter with $(V_1 \otimes V_2)(n^{ab})$, one sees that the claimed formula is a consequence of (A.3). □

C. On rectangles

Consider the Kronecker coefficients $G(m,n) := g(n^m, n^m, n^m)$ (see Table C.1).

REMARK C.1. (a) The first row of the table is trivial.

(b) For a proof that the pattern in the second row persists, see Corollary D.2.

(c) We have computed $G(3,n)$ (i.e., the quantities in the third row) for $n \leqslant 16$, and have found that the data is consistent with

$$\sum_{n \geqslant 0} G(3,n)\, q^n = \frac{1}{(1-q^2)(1-q^3)(1-q^4)}.$$

If true, then $G(3,n)$ is asymptotic to $n^2/48$, has a quasi-period of 12, and a partial fraction analysis of the right hand side would imply

$$G(3,n) = \begin{cases} [n(n+6)/48] & \text{if } n \text{ is odd,} \\ [(n+4)(n+8)/48] & \text{if } n \text{ is even,} \end{cases}$$

where $[x]$ denotes the integer nearest to x.

(d) The conjugation invariance $g(n^m, n^m, n^m) = g(m^n, m^n, n^m)$ and Proposition A.2 imply that $G(m,n)$ can be nonzero only for $m \leqslant n^2$.

(e) Noting that the $(n^2)^n$-complement of an $m\times n$ rectangle is an $(n^2-m)\times n$ rectangle, the complementation formula in Proposition B.1 implies

$$G(m,n) = g(n^{n^2-m}, n^{n^2-m}, (n^2-m)^n) = G(n^2-m,n). \tag{C.1}$$

That is, the n-th column of the matrix $[G(m,n)]$ is symmetric for $0 \leqslant m \leqslant n^2$.

Table C.1 and the symmetry (C.1) suggest that the distribution of $G(m,n)$ for fixed (large) n is bell-shaped. In particular, it seems likely that the vanishing of $G(2,n)$ and $G(n^2-n,n)$ for n odd are the only zeroes of $G(m,n)$ in the feasible range $0 \leqslant m \leqslant n^2$.

D. Kronecker coefficients and Gaussian coefficients

Let $p_k(m,n)$ denote the number of partitions of k whose Young diagrams fit inside an $m\times n$ rectangle. It is well known that these quantities are the coefficients of the Gaussian polynomials. More precisely, we have

$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \frac{(1-q^{n+m})\cdots(1-q^{n+1})}{(1-q^m)\cdots(1-q)} = \sum_k p_k(m,n)q^k.$$

Also well-known is that $p_k(m,n)$ is unimodal with respect to k, and there are many proofs, some quite elementary. The following result, noticed earlier by Vallejo [**V2**] and Pak and Panova [**PP**], shows that nonnegativity of Kronecker coefficients also implies unimodality.

Proposition D.1. *We have* $g(n^m, n^m, (mn-k,k)) = p_k(m,n) - p_{k-1}(m,n)$.

Proof. In the language of symmetric functions, we seek to evaluate $\langle s_\rho * s_{(mn-k,k)}, s_\rho\rangle$, where ρ denotes the rectangle partition n^m. By the Jacobi-Trudi identity one knows that

$$s_{(mn-k,k)} = h_{mn-k}h_k - h_{mn-k+1}h_{k-1},$$

so it suffices to show that $\langle s_\rho * h_{mn-k}h_k, s_\rho\rangle = p_k(m,n)$.

By the combinatorial rule for $s_\alpha * h_\beta$ (see Exercise 7.84.a in [**EC2**]), one knows that $s_\rho * h_{mn-k}h_k$ is the sum of $s_\alpha s_{\rho/\alpha}$, where α ranges over all partitions of size k contained in ρ. Moreover, since $\rho = n^m$ is a rectangle, it follows that ρ/α is the 180 degree rotation of the n^m-complement of α. One knows that skew Schur functions are invariant under rotations, so we have

$$s_\rho * h_{mn-k}h_k = \sum_{|\alpha|=k} s_\alpha s_{\tilde\alpha}, \quad \text{where } \tilde\alpha := n^m + \alpha^* = (n-\alpha_m,\ldots,n-\alpha_2,n-\alpha_1).$$

Recalling the basic adjoint relationship $\langle s_\mu s_\nu, s_\lambda\rangle = \langle s_\nu, s_{\lambda/\mu}\rangle$, we obtain

$$\langle s_\rho * h_{mn-k}h_k, s_\rho\rangle = \sum_{|\alpha|=k}\langle s_{\tilde\alpha}, s_{\rho/\alpha}\rangle = \sum_{|\alpha|=k}\langle s_{\tilde\alpha}, s_{\tilde\alpha}\rangle = p_k(m,n),$$

since the Schur functions are orthonormal. $\square$

As far as we are aware, there is no explicit positive combinatorial description known for the quantities $p_k(m,n) - p_{k-1}(m,n)$. However, one case where it is relatively easy to inject the partitions of $k-1$ into the partitions of k occurs when $k \leqslant n$. In this case, the leftmost empty column of the diagram is inside the rectangle, and so adding a column of length 1 is an injective map. The leftover partitions of k are those with no columns of length 1, so in the case $k = n$ we obtain

Corollary D.2. *We have*

$$\sum_{m \geqslant 0} g(n \cdot (1^m, 1^m, (m-1,1)))q^n = \frac{1}{(1-q^2)(1-q^3)\cdots(1-q^m)},$$

and in the special case $m = 2$,

$$g(n^2, n^2, n^2) = \begin{cases} 1 & \text{if } n \text{ even,} \\ 0 & \text{if } n \text{ odd.} \end{cases}$$

Considering the special case $m = 4$ of the above result, we see that the conjecture in Remark C.1(c) would imply the peculiar identity $g(n^4, n^4, (3n, n)) = g(n^3, n^3, n^3)$.

Remark D.3. Recall that the irreducible $\mathrm{sl}_2(\mathbb{C})$-module of highest weight m is the symmetric power $S^m(\mathbb{C}^2)$. It is easy to show that $p_k(m,n)$ is the dimension of the subspace of weight $mn - 2k$ in the plethystic composition $S^m(S^n(\mathbb{C}^2))$, so a further consequence of Proposition D.1 is that the Kronecker coefficient $g(n^m, n^m, (mn-k, k))$ is the multiplicity of $S^{mn-2k}(\mathbb{C}^2)$ in $S^m(S^n(\mathbb{C}^2))$.

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