Generalized Stability of Kronecker Coefficients

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Dedicated to Richard Stanley on the occasion of his 70th birthday.

Abstract. Kronecker coefficients are tensor product multiplicities for the irreducible representations of the symmetric group. In this paper, we identify directions in the parameter space for tensor products along which these multiplicities are monotone convergent, generalizing a classical result of Murnaghan.

Contents

1. Introduction

Let I_α denote the irreducible representation of S_m indexed by a partition α of m. Given a triple of partitions of m, say α, β, γ, the associated *Kronecker coefficient* is

$$g(\alpha\beta\gamma) = \text{multiplicity of } I_\alpha \text{ in } I_\beta \otimes I_\gamma = \dim(I_\alpha \otimes I_\beta \otimes I_\gamma)^{S_m}.$$

It is a major open problem to find a positive combinatorial formula for these multiplicities.

A well-known result of Murnaghan [**Mu**] asserts that if we grow the first (largest) parts in a triple of partitions, the Kronecker coefficient stabilizes. That is, the multiplicity

$$g(\alpha + n, \beta + n, \gamma + n)$$

is independent of n for n sufficiently large. One can also show that it is weakly increasing as a function of n.

Stability results of this type have long been a topic of interest in representation theory. For example, Schur-Weyl duality more or less explains the fact that tensor product multiplicities for $gl(V)$ depend only on the partitions associated to the highest weights involved, and not on the dimension of V. For more recent work, see for example the categorical approaches to stability phenomena in the symmetric groups and classical groups in the papers by Church, Ellenberg and Farb [**CEF**] and Sam and Snowden [**SS**].

Our goal in this paper is to show that Murnaghan's stability result can be vastly generalized—there are many lines in "triple partition space" along which Kronecker coefficients are monotone convergent. One thing we have not attempted to do here, although it would be an interesting follow-up project, is to express the stable limits arising as some natural representation-theoretic quantities. In this direction, it should be noted that for Murnaghan's stable limits, such interpretations are available. For example, Bowman, De Visscher and Orellana have shown recently that Murnaghan's limits are related to tensor product multiplicities in the partition algebra [**BDO**], and there is also a plethystic interpretation due to Brion (see Section 3.4 of [**Br**]).

In more detail, consider a Kronecker triple $\alpha\beta\gamma$; i.e., a triple of partitions such that the Kronecker coefficient $g(\alpha\beta\gamma)$ is positive. What we study in this paper are the conditions under which $\alpha\beta\gamma$ is "stable" in the sense that for all triples $\lambda\mu\nu$, the sequences

$$g(\lambda + n\alpha, \mu + n\beta, \nu + n\gamma) = g(\lambda\mu\nu + n \cdot \alpha\beta\gamma)$$

converge as $n \to \infty$. Such sequences are always monotone increasing (see Corollary 2.2), so in fact convergence is equivalent to being bounded. Note also that in this context, Murnaghan's stability result amounts to the statement that the triple $(1, 1, 1)$ is stable.

The methods we use to identify stable triples involve the analysis of integer points in polyhedra. This should not be surprising, since integer points and polyhedra have become commonplace in combinatorial representation theory. Guided by the intuition that it may be possible to describe Kronecker coefficients this way, we would expect that stretched Kronecker coefficients $g(n \cdot \alpha\beta\gamma)$ should be Ehrhart quasi-polynomials. This motivates our conjecture that a triple $\alpha\beta\gamma$ is stable if and only if $g(n \cdot \alpha\beta\gamma) = 1$ for $n \geqslant 1$. Indeed, this conjecture would follow from a hypothetical polyhedral description of Kronecker coefficients satisfying certain mild technical conditions. (See Section 4.)

Our main results are in Sections 6 and 7. We use polytopes whose integer points describe tensor product multiplicities for *permutation* representations of S_m to deduce the existence of stable Kronecker triples when these associated polytopes are 0-dimensional. (See Theorems 6.1 and 7.4.) This in turn leads to some interesting questions about contingency tables (the integer points of transportation polytopes), and some unexpected positivity results for Kronecker coefficients. These results are preceded in Section 5 by a similar but easier stability analysis for Kostka numbers (irreducible multiplicities for permutation representations) that is used in the proof of Theorem 6.1. In Section 9, we discuss stability in higher dimensions. To give a simple example that illustrates the phenomenon, it will develop that the Kronecker coefficient

$$g(\alpha\beta\gamma + m \cdot (2, 11, 11) + n \cdot (11, 2, 11))$$

is independent of m and n provided that both are sufficiently large.

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2. Monotonicity

For a partition α, let $V(\alpha)$ denote the irreducible $\mathrm{gl}(V)$-module with highest weight α. This makes sense as long as the dimension of V is at least the number of parts of α, and may be identified with 0 otherwise.

Recall that $V(m) = S^m(V)$ is the degree m part of the symmetric algebra of V.

Kronecker coefficients also arise in the representation theory of $\mathrm{gl}(V)$. For example, it is a corollary of Exercise 7.78.f in [**EC2**] that for a partition triple $\alpha\beta\gamma$ of size m,

$$g(\alpha\beta\gamma) = \text{multiplicity of } V_1(\alpha) \otimes V_2(\beta) \otimes V_3(\gamma) \text{ in } S^m(V_1 \otimes V_2 \otimes V_3) \tag{2.1}$$

as a $\mathrm{gl}(V_1) \oplus \mathrm{gl}(V_2) \oplus \mathrm{gl}(V_3)$-module, provided of course that V_1, V_2, V_3 have sufficiently large dimensions compared to the number of parts in α, β, γ (respectively).

The following fundamental property of Kronecker coefficients has been noted previously by Manivel (see the discussion on p. 157 of [**Ma**]).

Proposition 2.1. *If $g(\alpha\beta\gamma) > 0$, then $g(\lambda\mu\nu + \alpha\beta\gamma) \geqslant g(\lambda\mu\nu)$.*

Proof. Let $V = V_1 \oplus V_2 \oplus V_3$, and assume that the dimension of each space V_i is large. Given the description in (2.1), it follows that $g(\alpha\beta\gamma)$ is the dimension of the space of maximal vectors of weight $\alpha\oplus\beta\oplus\gamma$ in $S^m(V)$. (A weight vector is "maximal" if it is killed by the upper triangular subalgebra of $\mathrm{gl}(V_1)\oplus\mathrm{gl}(V_2)\oplus\mathrm{gl}(V_3)$.)

In the polynomial ring formed by the full symmetric algebra of V, the maximal vectors form a graded subring. In particular, if $g(\lambda\mu\nu)=r$ and $g(\alpha\beta\gamma)>0$, then there exist linearly independent maximal vectors $f_1,\dots,f_r\in S^n(V)$ of weight $\lambda\oplus\mu\oplus\nu$, and a (nonzero) maximal vector $g\in S^m(V)$ of weight $\alpha\oplus\beta\oplus\gamma$, where m and n denote the sizes of $\alpha\beta\gamma$ and $\lambda\mu\nu$. It follows that $f_1g,\dots,f_rg\in S^{m+n}(V)$ are linearly independent maximal vectors of weight $(\lambda+\alpha)\oplus(\mu+\beta)\oplus(\nu+\gamma)$. $\square$

An immediate corollary is the known fact that

$$\mathcal{G}:=\{\alpha\beta\gamma : g(\alpha\beta\gamma)>0\}$$

is a semigroup. We will refer to $\mathcal{G}$ as the *Kronecker semigroup*.

Corollary 2.2. *If $g(\alpha\beta\gamma)>0$, then $g(\lambda\mu\nu+n\cdot\alpha\beta\gamma)$ is a weakly increasing function of n. In particular, it converges if and only if it is bounded.*

Example 2.3. Since $g(11,11,11)=0$, we have no *a priori* guarantee that adding columns of length 2 to a triple of partitions will produce a monotone increasing sequence of Kronecker coefficients. In fact (see Remark 8.5), we have

$$g(n^2,n^2,n^2)=\begin{cases}1 & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd,}\end{cases}$$

so both monotonicity and convergence may fail. On the other hand, after checking that $g(22,22,22)=1$, Corollary 2.2 implies that the sequence $g(\lambda\mu\nu+n\cdot(11,11,11))$ may be split into a pair of monotone increasing subsequences for even and odd n. It also turns out that these subsequences converge, as we shall see in Section 8.

3. Some non-convergence

The following is a peculiar elementary fact about polynomials.

Lemma 3.1. *If f_1 and f_2 are linearly independent homogeneous polynomials of the same degree, then they are algebraically independent.*

Proof. Arguing by contradiction, we may suppose that there is a nontrivial dependence relation of the form $\sum a_i f_1^i f_2^{n-i}=0$ for some $n\geqslant 2$ and some scalars $a_0,\dots,a_n$. Among all such relations, choose one that minimizes n.

Without loss of generality, we may assume that f_1 and f_2 have no common factor, otherwise any such common factor p may be cancelled from both f_1 and f_2 and the dependence relation remains valid. Thus we may choose an irreducible factor p of f_1 that does not divide f_2. Since every term in the dependence relation except f_2^n is divisible by p, it follows that $a_0=0$. Thus every nonzero summand in the dependence relation carries a factor of f_1. Since we could delete this factor from all of the terms, we have contradicted the fact that we chose a dependence relation that minimized n. $\square$

Proposition 3.2. *If $g(\alpha\beta\gamma) \geqslant 2$, then $g(n \cdot \alpha\beta\gamma) \geqslant n+1$ for $n \geqslant 0$.*

Proof. Continuing the notation from the proof of Proposition 2.1, note that since $g(\alpha\beta\gamma) \geqslant 2$, we can find two linearly independent maximal vectors $f_1, f_2 \in S^m(V)$, where m is the common size of α, β, γ. By Lemma 3.1, it follows that the maximal vectors $f_1^n, f_1^{n-1}f_2, \dots, f_2^n$ in $S^{mn}(V)$ are linearly independent; i.e., $g(n \cdot \alpha\beta\gamma) \geqslant n+1$. □

It happens that $g(n \cdot (42, 42, 42)) = n + 1$, so this bound can be sharp.

In order to provide better lower bounds on the growth of "stretched" Kronecker coefficients $g(n \cdot \alpha\beta\gamma)$ when $g(\alpha\beta\gamma) > 2$, it would be interesting to generalize Lemma 3.1. More precisely, given linearly independent polynomials $f_1, \dots, f_r$ that are homogeneous of degree m, we would like to determine the minimum dimension of

$$\operatorname{Span}\left\{ f_1^{i_1} \cdots f_r^{i_r} : i_1 + \cdots + i_r = n \right\}, \tag{3.1}$$

over all possible $f_1, \dots, f_r$. Letting $\delta(n, m, r)$ denote this minimum, the same reasoning as above implies the following result.

Proposition 3.3. *If $\alpha\beta\gamma$ is a partition triple of size m and $g(\alpha\beta\gamma) = r$, then*

$$g(n \cdot \alpha\beta\gamma) \geqslant \delta(n, m, r).$$

Lemma 3.1 shows that $\delta(n, m, 2) = n + 1$, and $\delta(n, m, 1) = 1$ is trivial.

Remark 3.4. Deriving lower bounds for $\delta(n, m, r)$ when $r \geqslant 3$ seems difficult, but it is possible to guess where to look for extreme cases. For example, assuming $r \leqslant m + 1$, we could take $f_1, \dots, f_r$ to be monomials of degree m in two variables. The space in (3.1) would thus be a subspace of the monomials of degree mn in two variables; hence

$$\delta(n, m, r) \leqslant mn + 1 \quad \text{for } r \leqslant m + 1.$$

Similarly, if $r \leqslant \binom{m+2}{2}$, we could take $f_1, \dots, f_r$ to be monomials of degree m in three variables, and so on. In the specific case $r = 3$, $m \geqslant 2$, we could take $f_1 = x^m$, $f_2 = x^{m-1}y$, $f_3 = x^{m-2}y^2$; this yields the upper bound $\delta(n, m, 3) \leqslant 2n + 1$ for $m \geqslant 2$.

Does $\delta(n, m, r)$ grow quadratically with n when $r > m + 1$?

4. Stable triples

Given the preceding observations, it is natural to define the partition triple $\alpha\beta\gamma$ to be *stable* (or *bounded*) if $g(\alpha\beta\gamma) > 0$ and the sequence

$$\{g(\lambda\mu\nu + n \cdot \alpha\beta\gamma) : n \geqslant 0\} \tag{4.1}$$

is convergent (or equivalently, bounded) for all triples $\lambda\mu\nu$ such that $g(\lambda\mu\nu) > 0$.

Note that there is no harm in requiring $g(\lambda\mu\nu) > 0$; we could replace $\lambda\mu\nu$ with some $\lambda\mu\nu + k \cdot \alpha\beta\gamma$ if necessary unless the sequence $g(\lambda\mu\nu + n \cdot \alpha\beta\gamma)$ is identically 0.

Plausibly, we could also investigate stable triples $\alpha\beta\gamma$ with $g(\alpha\beta\gamma) = 0$. However, we have previously noticed that the sequences (4.1) need not be monotone and need not converge in such cases. Given that $g(k \cdot \alpha\beta\gamma) > 0$ for some k, it seems that the right question to ask in that case is whether $k \cdot \alpha\beta\gamma$ is a stable triple.

Problem 4.1. *Characterize the stable triples in some practical way.*

Proposition 3.2 shows that if $\alpha\beta\gamma$ is a stable triple, then $g(\alpha\beta\gamma) = 1$.

Unfortunately, the converse fails. For example, $g(2^3, 2^3, 2^3) = 1$ but $(2^3, 2^3, 2^3)$ is not a stable triple, since $g(4^3, 4^3, 4^3) = 2$ and therefore $g((4n)^3, (4n)^3, (4n)^3) \geqslant n + 1$ by Proposition 3.2. Thus a stronger necessary condition is

Proposition 4.2. *If the triple $\alpha\beta\gamma$ is stable, then $g(n \cdot \alpha\beta\gamma) = 1$ for $n \geqslant 1$.*

Conjecture 4.3. *Conversely, if $g(n \cdot \alpha\beta\gamma) = 1$ for $n \geqslant 1$, then $\alpha\beta\gamma$ is stable.*

A proof of this conjecture might not resolve Problem 4.1, but at least it would reduce the analysis to one sequence per triple. More optimistically, if a positive combinatorial description of the Kronecker coefficients is found, it is quite plausible that $g(\alpha\beta\gamma)$ will turn out to be expressible as the number of integer points in a polytope $P_{\alpha\beta\gamma}$. (Certainly this can be done for Littlewood-Richardson coefficients; see [**BZ2**] for possibly the first such description.) Assuming the polytope scales linearly with $\alpha\beta\gamma$ (i.e., $nP_{\alpha\beta\gamma} = P_{n\cdot\alpha\beta\gamma}$), one could interpret $g(n \cdot \alpha\beta\gamma)$ as the Ehrhart quasi-polynomial associated to $P_{\alpha\beta\gamma}$; having $g(n \cdot \alpha\beta\gamma) = 1$ for all n amounts to $P_{\alpha\beta\gamma}$ being 0-dimensional and consisting of a single integer point. This is of course a finite condition, testable via linear programming.

Continuing in this optimistic vein, the following result shows that if we demand slightly more from our hypothetical polytope $P_{\alpha\beta\gamma}$, being 0-dimensional would be equivalent to stability and Conjecture 4.3 would be true.

Proposition 4.4. *Fix rational matrices A and B of appropriate sizes, and for $y \in \mathbb{Z}^l$ let $f(y)$ denote the number of integer points in the rational polyhedron*

$$P(y) = \{x \in \mathbb{R}^k : Ax \leqslant By\}. \tag{4.2}$$

Assuming $P(y)$ is non-empty, the sequence $\{f(z + ny) : n \geqslant 0\}$ is bounded for all $z \in \mathbb{Z}^l$ if and only if $P(y)$ is 0-dimensional.

Proof. Assume $P(y)$ has dimension d. Since $f(ny)$ is the Ehrhart quasi-polynomial associated to $P(y)$, it necessarily grows at a rate asymptotically proportional to n^d (or is infinite for some values of n, if $P(y)$ is unbounded). Thus the condition $d = 0$ is necessary.

For the converse, we may suppose that $P(y)$ consists of a single rational vertex p. By linear programming duality, one knows that for (say) the i-th coordinate, there must exist a nonnegative linear combination of the inequalities defining $P(y)$ that prove $x_i \geqslant p_i$ throughout the polytope. Similarly, there must be another such combination that proves $x_i \leqslant p_i$. Since the defining inequalities vary linearly with y, it follows that if we take those same combinations and apply them to the inequalities that define $P(z+ny)$, we will necessarily obtain inequalities that prove

$$np_i + a \leqslant x_i \leqslant np_i + b \quad \text{for all } x \in P(z+ny),$$

where a and b are scalars independent of n. Thus there is an upper bound on the possible number of distinct integer values for each coordinate of each point in $P(z+ny)$. $\square$

By analyzing integer points and polyhedra indirectly related to Kronecker coefficients, we will show that the above result can be used to identify many stable triples.

5. Stability of Kostka numbers

Recall that for partitions α and β of m, the Kostka number $K_{\alpha,\beta}$ is the dimension of the β-weight subspace of $V(\alpha)$. It is also the multiplicity of I_α in the permutation representation of S_m induced by the Young subgroup $S_{\beta_1} \times S_{\beta_2} \times \cdots$.

It is well known that $K_{\alpha,\beta}$ may be described combinatorially as the number of semistandard tableaux of shape α and content β. These are the positive integer arrays

$$[T_{ij}]_{i \geqslant 1, 1 \leqslant j \leqslant \alpha_i}$$

satisfying $T_{i,j} \leqslant T_{i,j+1}$ and $T_{i,j} < T_{i+1,j}$ (i.e., weakly increasing rows and strictly increasing columns) such that the number of entries equal to k is β_k, for $k \geqslant 1$.

Recall that $\alpha \geqslant \beta$ in dominance order if $\alpha_1 + \cdots + \alpha_k \geqslant \beta_1 + \cdots + \beta_k$ for $1 \leqslant k < \ell(\alpha)$.

Proposition 5.1. *Let α, β, and γ be partitions of m.*

(a) (*Well known.*) *If $\beta \leqslant \gamma$, then $K_{\alpha,\beta} \geqslant K_{\alpha,\gamma}$.*
(b) (*Well known.*) *We have $K_{\alpha,\beta} > 0$ if and only if $\alpha \geqslant \beta$.*
(c) *If $\alpha \geqslant \beta$, then $K_{\lambda+\alpha,\mu+\beta} \geqslant K_{\lambda,\mu}$ for all λ, μ.*

Proof. (a) See Example I.7.9 in [**M**].

(b) If there is a semistandard tableau of shape α and content β, then the $\beta_1 + \cdots + \beta_k$ entries $\leqslant k$ must fit into the $\alpha_1 + \cdots + \alpha_k$ positions in the first k rows; i.e., $\alpha \geqslant \beta$. Conversely, if $\alpha \geqslant \beta$, then (a) implies $K_{\alpha,\beta} \geqslant K_{\alpha,\alpha} = 1$.

(c) We have $K_{\lambda+\alpha,\mu+\beta} \geqslant K_{\lambda+\alpha,\mu+\alpha} \geqslant K_{\lambda,\mu}$, the first inequality being a consequence of (a), and the second being a consequence of the fact that we can take a semistandard tableau of shape λ and content μ, shift the i-th row α_i columns to the right, and assign i to each of the vacated positions. The result will be a semistandard tableau of shape $\lambda+\alpha$ and content $\mu+\alpha$. $\square$

Thus the pairs (α,β) indexing positive Kostka numbers form a semigroup, and these numbers are monotone increasing along any affine ray in a direction inside the semigroup. By analogy with the Kronecker semigroup, we define (α,β) to be *Kostka-stable* if $K_{\alpha,\beta}>0$ and the sequences $\{K_{\lambda+n\alpha,\mu+n\beta} : n \geqslant 0\}$ are convergent (or equivalently, bounded) for all pairs (λ,μ). Characterizing the Kostka-stable pairs will turn out to be more than just a warm-up exercise—it will help us identify Kronecker-stable triples.

PROPOSITION 5.2. *The pair (α,β) is Kostka-stable if and only if $K_{\alpha,\beta}=1$.*

As a first step towards proving the above result, it will be helpful to give a combinatorial characterization of the pairs (α,β) such that $K_{\alpha,\beta}=1$. In fact this has been done previously for all semisimple Lie algebras by Berenstein and Zelevinsky [**BZ1**], so the following description should be seen as a special case of their work.

Of course we know that if $K_{\alpha,\beta}=1$ then $\alpha \geqslant \beta$. If it happens that one of the defining inequalities for dominance is tight, say $\alpha_1+\cdots+\alpha_k=\beta_1+\cdots+\beta_k$, then all of the entries $\leqslant k$ in a semistandard tableau of shape α and content β must fill the first k rows, and the entries $>k$ must fill the remaining rows. It follows that the subtableau formed by the first k rows and the subtableau formed by all other rows may be specified independently of each other, so we have

$$K_{\alpha,\beta}=K_{\alpha^{(1)},\beta^{(1)}}\cdot K_{\alpha^{(2)},\beta^{(2)}}, \tag{5.1}$$

where $\alpha^{(1)}=(\alpha_1,\ldots,\alpha_k)$, $\alpha^{(2)}=(\alpha_{k+1},\alpha_{k+2},\ldots)$, and $\beta^{(1)},\beta^{(2)}$ are defined similarly. In this situation we say that the pair (α,β) *factors* into $(\alpha^{(1)},\beta^{(1)})$ and $(\alpha^{(2)},\beta^{(2)})$. By iteration, one may factor (α,β) further into a set of *primitive* pairs; i.e., pairs of partitions related in dominance order such that all of the defining inequalities are strict.

The pair $(\alpha,\beta)=(m,m)$ is a somewhat degenerate case, but still primitive. Indeed, there are only $\ell(\alpha)-1$ defining inequalities for dominance, and hence none in this case.

We will say that the pair (α,β) has *shape* α and will count 0 as a part of α having multiplicity $\ell(\beta)-\ell(\alpha)$. Given that $\alpha \geqslant \beta$, this multiplicity is nonnegative.

LEMMA 5.3 [**BZ1**]. *We have $K_{\alpha,\beta}=1$ if and only if $\alpha \geqslant \beta$ and every primitive factor of (α,β) has a shape with at most two distinct part sizes, one of which occurs only once.*

Proof. By (5.1) and Proposition 5.1(b), we may assume (α,β) is primitive and $\alpha \geqslant \beta$.

If it happens that $\alpha=\beta$, then primitivity forces $\alpha=\beta=(m)$ for some m, and it is clear that $K_{\alpha,\beta}=1$. Thus we assume henceforth that α strictly dominates β.

Suppose $p>q>r \geqslant 0$ are three distinct part sizes occurring in α. We may assume further that q is the only part size in the range between p and r. Thus α has a run of consecutive parts of the form $(p,q,\ldots,q,r)$. Letting α^- denote the partition obtained from α by decreasing the last p by 1 and increasing the first r by 1, the primitivity of the pair (α, β) implies

$$\alpha > \alpha^- \geqslant \beta,$$

from which we claim it follows that

$$K_{\alpha,\beta} \geqslant K_{\alpha,\alpha^-} = K_{(p,q,\dots,q,r),(p-1,q,\dots,q,r+1)} \geqslant 2.$$

Indeed, the first inequality is a consequence of Proposition 5.1(a), the equality follows from (5.1) and the fact that $((p, q, \dots, q, r), (p-1, q, \dots, q, r+1))$ is the only nontrivial primitive factor of the pair (α, α^-), and the last inequality is best understood by examining the following pair of semistandard tableaux of shape 3221 and content 2222:

$$\begin{array}{ccc} 1 & 1 & 4 \\ 2 & 2 & \\ 3 & 3 & \\ 4 & & \end{array} \qquad \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 2 & \\ 3 & 4 & \\ 4. & & \end{array}$$

More general shapes of the form $(p, q, \dots, q, r)$ can be handled similarly.

Now suppose that α has only two part sizes $p > q \geqslant 0$. If $p - q = 1$, then the only partitions strictly dominated by α all have greater length. This means that 0 must be a part of α; i.e., $p = 1$, $q = 0$, and α is minimal in dominance order—a contradiction.

If p and q both occur at least twice and $p - q > 1$, we will obtain a partition α^- by decreasing the last two p's and increasing the first two q's by 1 each. Furthermore, since (α, β) is primitive and the largest parts of α are all p, the largest parts of β are at most $p - 1$ and the sum $\alpha_1 + \cdots + \alpha_k$ will exceed the corresponding sum for β by at least k as long as $\alpha_k = p$. Hence,

$$\alpha > \alpha^- \geqslant \beta,$$

and by reasoning similar to the previous case we obtain

$$K_{\alpha,\beta} \geqslant K_{\alpha,\alpha^-} = K_{(p,p,q,q),(p-1,p-1,q+1,q+1)} \geqslant 2.$$

In particular, the last inequality can be understood by examining the following pair of semistandard tableaux of shape 4422 and content 3333:

$$\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 4 \\ 3 & 3 & & \\ 4 & 4 & & \end{array} \qquad \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 4 \\ 3 & 3 & & \\ 4 & 4. & & \end{array}$$

The remaining possibilities are $\alpha = pq^{l-1}$, $p^{l-1}q$, or p^l, where $p > q \geqslant 0$ and $l = \ell(\beta)$.

In the first case, the q columns of length l in a semistandard tableau of shape α and content β may be filled in only one way—each of the entries $1, 2, \dots, l$ must appear once each. The remaining columns all have length 1 and the entries in those columns must appear in sorted order; i.e., $K_{\alpha,\beta} = 1$. In the second case, the columns of length less than l all have length $l-1$, and so must omit exactly one element each from the set $\{1, 2, \ldots, l\}$. The fact that the rows of a semistandard tableau must increase weakly forces these columns to be sorted in lexicographic order and again $K_{\alpha,\beta} = 1$.

In case $\alpha = p^l$, there are no partitions of length l strictly dominated by α. □

Since factoring (α, β) into primitive pairs commutes with rescaling, we obtain

COROLLARY 5.4. *If $K_{n\alpha,n\beta} = 1$ for some $n \geqslant 1$, then $K_{n\alpha,n\beta} = 1$ for all $n \geqslant 1$.*

Proof of Proposition 5.2. Semistandard tableaux are in bijection with Gelfand patterns. More precisely, given a tableau T of shape α and content β, set $l = \ell(\beta)$ and let x_{ij} denote the number of entries in row i of T that are $\leqslant j$. The triangular array $x_{i,j} : 1 \leqslant i \leqslant j \leqslant l$ uniquely determines T, and moreover, these arrays are characterized by the relations

$$x_{i,j+1} \geqslant x_{i,j} \geqslant x_{i+1,j+1}, \quad x_{i,l} = \alpha_i, \quad \sum_i x_{i,j} = \beta_1 + \cdots + \beta_j.$$

Thus one sees that there is a "Gelfand polytope" $P_{\alpha,\beta}$ whose integer points count the semistandard tableaux of shape α and content β. Since this family of polytopes clearly has the form of (4.2), Proposition 4.4 implies that (α, β) is stable if and only if $\alpha \geqslant \beta$ and $P_{\alpha,\beta}$ is 0-dimensional, or equivalently, if and only if $P_{n\alpha,n\beta}$ contains exactly one integer point for all $n \geqslant 1$. By Corollary 5.4, this is equivalent to the condition $K_{\alpha,\beta} = 1$. □

REMARK 5.5. It would be interesting to give a direct proof of Proposition 5.2 that bypasses the explicit description of the pairs (α, β) such that $K_{\alpha,\beta} = 1$. For example, if one could prove that the Gelfand polytope $P_{\alpha,\beta}$ always has integer vertices then Proposition 5.2 would be a direct corollary of Proposition 4.4. However, De Loera and McAllister have shown that $P_{\alpha,\beta}$ does not always have integer vertices, and the smallest counterexamples involve Gelfand patterns with at least 5 rows [**DM**]. In this light, Corollary 5.4 can be seen as a proof of a much weaker statement; namely, that if the Gelfand polytope contains only one integer point, then it must be 0-dimensional.

6. Transportation polytopes and stability

Given a triple of partitions $\alpha\beta\gamma$ of size m, let us define

$$h(\alpha\beta\gamma) = \text{multiplicity of } I_\gamma \text{ in } M_\alpha \otimes M_\beta = \dim(M_\alpha \otimes M_\beta \otimes I_\gamma)^{S_m}.$$

where M_α and M_β denote the permutation representations of S_m induced by Young subgroups of type α and β. Of course $M_\alpha \otimes M_\beta$ is also a permutation representation of S_m, and it is not hard to show that its orbits are indexed by the set $C(\alpha, \beta)$ of "contingency tables" of type (α, β); i.e., nonnegative integer matrices with row sum vector α and column sum vector β. Moreover, the S_m-orbit indexed by a table $T \in C(\alpha, \beta)$ carries an action isomorphic to $M_{\mathrm{co}(T)}$, where $\mathrm{co}(T)$ (the *content* of T) denotes the partition of m formed by the entries of T. Thus we have

$$M_\alpha \otimes M_\beta \cong \bigoplus_{T\in C(\alpha,\beta)} M_{\mathrm{co}(T)}.$$

This is equivalent to Exercise 7.84.b of [**EC2**] or Example I.7.23(3) of [**M**].

We remark that $C(\alpha,\beta)$ is the set of integer points of the transportation polytope

$$Q(\alpha,\beta) = \left\{ [x_{ij}] : x_{ij} \geqslant 0,\ \textstyle\sum_j x_{ij} = \alpha_i,\ \sum_i x_{ij} = \beta_j \right\}.$$

Also, recalling that the Kostka number $K_{\gamma,\alpha}$ is the multiplicity of I_γ in M_α, we have

$$h(\alpha\beta\gamma) = \sum_{T\in C(\alpha,\beta)} K_{\gamma,\mathrm{co}(T)}. \tag{6.1}$$

Combining this with the Gelfand pattern polytope discussed earlier, one may easily construct a polytope whose integer points are counted by $h(\alpha\beta\gamma)$ (see Remark 7.3). However, it will be easier to identify stable Kronecker triples by analyzing a family of smaller polytopes that factors out the dependence of (6.1) on Kostka numbers.

Considering that $K_{\gamma,\mathrm{co}(T)} = 0$ unless $\gamma \geqslant \mathrm{co}(T)$, we define $Q(\alpha,\beta;\gamma)$ to be the polytope consisting of all matrices $X = [x_{ij}] \in Q(\alpha,\beta)$ whose "content" is dominated by γ. That is, the sum of any set of k entries of X must be $\leqslant \gamma_1 + \cdots + \gamma_k$ for all $k \geqslant 1$. The set $C(\alpha,\beta;\gamma)$ of integer points in $Q(\alpha,\beta;\gamma)$ is thus the set of contingency tables in $C(\alpha,\beta)$ whose contents are dominated by γ.

THEOREM 6.1. *We have $h(n\cdot\alpha\beta\gamma) = 1$ for all $n \geqslant 1$ if and only if the polytope $Q(\alpha,\beta;\gamma)$ is 0-dimensional, consisting of a single integer table T such that $K_{\gamma,\mathrm{co}(T)} = 1$. Furthermore, if either of these equivalent conditions hold, then*

(a) *there are unique partitions $\alpha^+ \geqslant \alpha$ and $\beta^+ \geqslant \beta$ such that $g(\alpha^+\beta^+\gamma) > 0$,*
(b) *the triple $\alpha^+\beta^+\gamma$ is stable, and*
(c) *we have $h(n\cdot\alpha^+\beta^+\gamma) = 1$ for all $n \geqslant 1$.*

If we know *a priori* that $g(\alpha\beta\gamma) > 0$, then of course $\alpha^+ = \alpha$ and $\beta^+ = \beta$. Hence,

COROLLARY 6.2. *If $g(\alpha\beta\gamma) > 0$ and $h(n\cdot\alpha\beta\gamma) = 1$ for all $n \geqslant 1$, then $\alpha\beta\gamma$ is stable.*

It should be noted that part (c) of Theorem 6.1 shows that $\alpha^+\beta^+\gamma$ satisfies the same hypotheses as $\alpha\beta\gamma$, so the sets of triples that can be proved stable as a consequence of either Theorem 6.1(b) or Corollary 6.2 are exactly the same.

Proof of Theorem 6.1. If $Q(\alpha, \beta; \gamma)$ consists of a single vertex $T \in C(\alpha, \beta; \gamma)$ and $K_{\gamma, \mathrm{co}(T)} = 1$, then nT is the unique vertex of the scaled polytope $Q(n\alpha, n\beta; n\gamma)$. Hence

$$h(n \cdot \alpha\beta\gamma) = K_{n\gamma, \mathrm{co}(nT)} = 1$$

by (6.1) and Corollary 5.4. Conversely, if $h(\alpha\beta\gamma) = 1$ then there can be only one positive summand in (6.1), and hence only one contingency table $T \in C(\alpha, \beta; \gamma)$, by Proposition 5.1(b). Moreover, this table T must satisfy $K_{\gamma, \mathrm{co}(T)} = 1$. If $Q(\alpha, \beta; \gamma)$ failed to be 0-dimensional, then some rescaling $Q(n\alpha, n\beta; n\gamma)$ would have two or more integer points, and hence $h(n \cdot \alpha\beta\gamma) > 1$ for some n. Thus, the two stated conditions are equivalent.

Given that the two equivalent conditions hold, consider that I_{α^+} and I_{β^+} occur with positive multiplicity in M_α and M_β whenever $\alpha^+ \geqslant \alpha$ and $\beta^+ \geqslant \beta$ (Proposition 5.1(b)). It follows that there can be only one such pair with $g(\alpha^+\beta^+\gamma) > 0$, or else we contradict the hypothesis that $h(\alpha\beta\gamma) = 1$.

To prove (b), we seek to show that $g(\lambda\mu\nu + n \cdot \alpha^+\beta^+\gamma)$ is bounded for any triple $\lambda\mu\nu$. Now since $I_{\lambda+n\alpha^+}$ and $I_{\mu+n\beta^+}$ are summands of $M_{\lambda+n\alpha}$ and $M_{\mu+n\beta}$, we have

$$g(\lambda\mu\nu + n \cdot \alpha^+\beta^+\gamma) \leqslant h(\lambda\mu\nu + n \cdot \alpha\beta\gamma) = \sum_{T \in C_n} K_{\nu+n\gamma, co(T)}, \tag{6.2}$$

where $C_n = C(\lambda + n\alpha,\, \mu + n\beta;\, \nu + n\gamma)$. Since the polytope $Q(\alpha, \beta; \gamma)$ has the form of (4.2) and is 0-dimensional by hypothesis, we may deduce from Proposition 4.4 that the polytope $Q_n = Q(\lambda + n\alpha,\, \mu + n\beta;\, \nu + n\gamma)$ has a bounded number of integer points. Moreover, from the proof one sees all of these integer points (i.e., members of C_n) have the form $T = nT_0 + E$, where T_0 is the unique contingency table in $C(\alpha, \beta; \gamma)$ and E is limited to a finite set of integer matrices. While it is possible that some of the matrices E have negative entries, all members of C_n are nonnegative, so by making a change of variable $n \to n + n_0$, with n_0 large enough so that $n_0T_0 + E$ is nonnegative, one may assume that each matrix E is nonnegative. Thus in (6.2) there are only finitely many summands, each of the form $K_{\nu+n\gamma, \varepsilon+\mathrm{co}(nT_0)}$ for various partitions ε. Proposition 5.2 and the hypothesis that $K_{\gamma, \mathrm{co}(T_0)} = 1$ imply that each of these summands is bounded.

For (c), note that Proposition 5.1(a) implies that $M_{n\alpha^+}$ may be embedded in $M_{n\alpha}$ as a submodule. Similarly, $M_{n\beta^+}$ embeds in $M_{n\beta}$, hence

$$1 = h(n \cdot \alpha\beta\gamma) \geqslant h(n \cdot \alpha^+\beta^+\gamma) \geqslant g(n \cdot \alpha^+\beta^+\gamma) = 1,$$

the last equality being a consequence of Proposition 4.2. □

Example 6.3. (a) All triples of the form (m, β, β) are stable. Indeed, if $\alpha = (m)$ then $M_{n\alpha}$ is the trivial representation of S_{nm}. Recalling that there is one copy of I_β in M_β, one sees from the definition that $h(nm, n\beta, n\beta) = 1$ for all $n \geqslant 1$. Apply Theorem 6.1.

(b) All triples of the form $(\alpha, \alpha', 1^m)$, where α' denotes the conjugate of α, are stable. Indeed, recalling that tensoring by the sign representation corresponds to conjugation of partitions, it is clear that $g(\alpha, \alpha', 1^m) = 1$. Moreover, since conjugation is order-reversing with respect to dominance, it follows that $M_{\alpha'} \otimes I_{1^m}$ contains one copy of I_α, and all other irreducible summands are lower in dominance order. Hence $h(\alpha, \alpha', 1^m) = 1$, so there is a unique contingency table in $C(\alpha, \alpha')$ with content dominated by 1^m. This table is the 0, 1-matrix T such that

$$T_{ij} = 1 \quad \text{if } 1 \leqslant j \leqslant \alpha_i,\ i \geqslant 1.$$

(In fact this is a restatement of the Gale-Ryser Theorem.) We further claim that the polytope $Q(\alpha, \alpha'; 1^m)$ is 0-dimensional. If not, it would contain $T + \varepsilon$ for some small nonzero matrix ε with row and column sums equal to 0. For such a matrix, we must have $\varepsilon_{ij} < 0$ only if $T_{ij} = 1$ and $\varepsilon_{ij} > 0$ only if $T_{ij} = 0$, otherwise, 1^m would not dominate the content of $T + \varepsilon$. Now consider the northeasternmost nonzero entry of ε; say it is in position (i, j). If $\varepsilon_{ij} < 0$, then any zeros of T in row i must also be zeros of ε, and ε would have a negative row sum. If $\varepsilon_{ij} > 0$, then any ones of T in column j must be zeros of ε and ε would have a positive column sum. Either way, we have a contradiction, proving the claim. The stability result now follows from Theorem 6.1.

We remark that recent work on Kronecker stability by Vallejo [**V**] (see Theorem 10.2) and Pak and Panova [**PP**] (see Theorem 1.1) provide alternate proofs that the triple $(m, 1^m, 1^m)$ is stable. Vallejo's result also provides a bound on the point at which the stable limit is reached.

THEOREM 6.4. *Let $T \in C(\alpha, \beta)$ be a contingency table with content γ. If there are no other tables in $C(\alpha, \beta)$ with content $\leqslant \gamma$, then T is a plane partition* (*i.e., it has weakly decreasing rows and columns*) *and $g(\alpha\beta\gamma) = 1$.*

Proof. Suppose T has two adjacent entries in the same row or column that are not in weakly decreasing order, say $T_{i,j} < T_{i+1,j}$. Since the sum of row i in T is at least as large as row $i+1$, there must also be a pair of entries in the same two rows that are in strict order, say $T_{i,k} > T_{i+1,k}$. If we decrease $T_{i+1,j}$ and increase $T_{i,j}$ by 1 each, the content partition will either decrease in dominance order or (if $T_{i+1,j} - T_{i,j} = 1$) stay unchanged. Similar remarks apply if we increase $T_{i+1,k}$ and decrease $T_{i,k}$ by 1. If we apply both operations, we preserve all row and column sums and contradict the uniqueness of T. Thus T must be a plane partition.

Since $K_{\gamma,\gamma} = 1$, it is clear from (6.1) that $1 = h(\alpha\beta\gamma) \geqslant g(\alpha\beta\gamma)$. Arguing by contradiction, we may suppose that $g(\alpha\beta\gamma) = 0$. In that case, there must be a pair of partitions $\alpha^+ \geqslant \alpha$ and $\beta^+ \geqslant \beta$ such that $g(\alpha^+\beta^+\gamma) > 0$, and hence $h(\alpha^+\beta^+\gamma) > 0$. Hence there is a contingency table $T' \in C(\alpha^+, \beta^+)$ with content $\leqslant \gamma$. Furthermore, either $\alpha^+ > \alpha$ or $\beta^+ > \beta$ (possibly both). Assuming the former, it must be possible to decrease (say) the i-th part of α^+ and increase the j-th part by 1 (for some $i < j$), obtaining a partition

δ such that $\alpha^+ > \delta \geqslant \alpha$. Since this requires $\alpha_i^+ > \alpha_j^+$, there must be a column k such that $T'_{ik} > T'_{jk}$. If we decrease T'_{ik} and increase T'_{jk} by 1, the result is a contingency table $T'' \in C(\delta, \beta^+)$ with $\mathrm{co}(T'') \leqslant \mathrm{co}(T') \leqslant \gamma$. Continuing in this way will generate a sequence of contingency tables $T', T'', T''', \ldots$ with weakly decreasing contents that terminates when it reaches T, the unique table in $C(\alpha, \beta)$ with content $\leqslant \gamma$. Since T has content γ, every table in this sequence must have content γ. However, content preservation occurs only if the pairs of entries being changed differ by 1. Each such step terminates with the changed pair in increasing order, contradicting the fact that T must be a plane partition. $\square$

COROLLARY 6.5. *If $\alpha\beta\gamma$ is a triple that satisfies the hypotheses of Theorem* 6.1, *and the unique contingency table in $C(\alpha, \beta; \gamma)$ has content γ, then $\alpha\beta\gamma$ is stable.*

EXAMPLE 6.6. If δ is a partition of length at most 4, with $\alpha = (\delta_1 + \delta_2, \delta_3 + \delta_4)$ and $\beta = (\delta_1 + \delta_3, \delta_2 + \delta_4)$, then $\alpha\beta\delta$ is stable and $g(\alpha\beta\delta) = 1$. Indeed, the transportation polytope $Q(\alpha, \beta)$ is 1-dimensional in this case, consisting of all 2×2 matrices of the form

$$X(t) = \begin{bmatrix} \delta_1 + t & \delta_2 - t \\ \delta_3 - t & \delta_4 + t \end{bmatrix}, \quad -\delta_4 \leqslant t \leqslant \delta_3.$$

In order for δ to dominate the content of $X(t)$, every entry, including $\delta_1 + t$ and $\delta_4 + t$, must be bounded above by δ_1 (hence $t \leqslant 0$) and below by δ_4 (hence $t \geqslant 0$). Thus the polytope $Q(\alpha, \beta; \delta)$ is 0-dimensional and the claim follows from Corollary 6.5.

Although it is well-known that all transportation polytopes $Q(\alpha, \beta)$ have integer vertices, this is not true for the polytopes $Q(\alpha, \beta; \gamma)$. Indeed, if we set $\delta = 1^4$ in the above example, one sees that $\mathrm{co}(X(t)) \leqslant 211$ precisely when $-1/2 \leqslant t \leqslant 1/2$. Thus even though $Q(22, 22; 211)$ contains only one integer point $X(0)$, it is not a lattice polytope and is not 0-dimensional. Note also that $X(0)$ has content 1^4 and $h(22, 22, 211) = K_{211,1^4} = 3$.

On the other hand, we have evidence in support of the following (cf. the analogous discussion for Gelfand polytopes in Remark 5.5):

CONJECTURE 6.7. *If $h(\alpha\beta\gamma) = 1$ (or equivalently, there is a unique contingency table T in $C(\alpha, \beta; \gamma)$ and this table satisfies $K_{\gamma,\mathrm{co}(T)} = 1$), then $Q(\alpha, \beta; \gamma)$ is a 0-dimensional polytope. In particular, there are unique partitions $\alpha^+ \geqslant \alpha$ and $\beta^+ \geqslant \beta$ such that $\alpha^+\beta^+\gamma$ is stable, and if the table T has content γ, then $\alpha^+ = \alpha$ and $\beta^+ = \beta$.*

Let us define a contingency table $T \in C(\alpha, \beta)$ to be *dominance-extremal* (or simply *extremal*) if there are no other contingency tables in $C(\alpha, \beta)$ with content $\leqslant \mathrm{co}(T)$.

PROBLEM 6.8. *Find a practical characterization of all extremal contingency tables.*

A solution of this problem could lead to a resolution of Conjecture 6.7. For example, if

$$\mathcal{L} := \{(\alpha, \beta, \mathrm{co}(T)) : T \text{ is extremal in } C(\alpha, \beta)\}$$

turns out to be invariant under rescaling (i.e., $\alpha\beta\gamma \in \mathcal{L}$ implies $n \cdot \alpha\beta\gamma \in \mathcal{L}$), this would imply Conjecture 6.7 whenever the unique contingency table in $C(\alpha, \beta; \gamma)$ has content γ. In any case, if $C(\alpha, \beta; \gamma)$ contains only one table, it is necessarily extremal.

Theorem 6.4 shows that extremal contingency tables are necessarily plane partitions. However, the converse is false in general. For example, consider the tables

$$\begin{matrix} 4 & 2 & 2 \\ 4 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \qquad \begin{matrix} 4 & 3 & 1 \\ 3 & 1 & 1 \\ 2 & 0 & 0 \end{matrix} \qquad \begin{matrix} 3 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 0. \end{matrix}$$

The first two are plane partitions and members of $C(852, 942)$, but the content of the first dominates the second, so it is not extremal. The third is a plane partition in $C(742, 742)$, but it is not symmetric. Thus it cannot be extremal, since its transpose is a table with the same content. In fact the content of this table turns out to be minimal in dominance order among all tables in $C(742, 742)$, so this example shows that having dominance-minimal content does not imply extremality.

PROPOSITION 6.9. *If T is a plane partition with two rows or two columns, then T is extremal. Furthermore, if T has row sum vector α, column sum vector β, and content γ, then $\alpha\beta\gamma$ is stable and $g(\alpha\beta\gamma) = 1$.*

Proof. Let $T = [T_{ij}]$ be a $2 \times n$ plane partition and ε a nonzero matrix such that (1) $T + \varepsilon$ has the same row and column sums as T, and (2) $\mathrm{co}(T + \varepsilon) \leqslant \mathrm{co}(T)$. The first condition implies that ε has row and column sums equal to zero, and the second implies that for every set of k entries of T whose sum matches the sum of the largest k entries of T, the corresponding entries of ε must have a non-positive sum.

Let ε_i denote the $(1, i)$-entry of ε. The $(2, i)$-entry is $-\varepsilon_i$, and $\varepsilon_1 + \cdots + \varepsilon_n = 0$.

We claim that for every $i \leqslant n$, there is a $j \leqslant i$ such that $\varepsilon_j + \varepsilon_{j+1} + \cdots + \varepsilon_i \leqslant 0$. Indeed, consider the set of all entries of T that are greater than $T_{1,i}$, together with the entry $T_{1,i}$. Since T is a plane partition, this involves both entries of columns 1 through $j - 1$ and the set of first-row entries in columns j through i for some $j \leqslant i$, and no other entries. The corresponding entries of ε sum to $\varepsilon_j + \cdots + \varepsilon_i$, since the two-entry columns sum to 0. This sum must be non-positive, so the claim follows.

Iterating this claim, we must also have $\varepsilon_k + \cdots + \varepsilon_{j-1} \leqslant 0$ for some $k < j$, and so on. Adding these inequalities together, we conclude that $\varepsilon_1 + \cdots + \varepsilon_i \leqslant 0$ for all i.

Now let i be the largest index such that $\varepsilon_i \neq 0$. Since $\varepsilon_1 + \cdots + \varepsilon_{i-1} \leqslant 0$ (by the previous claim) and $\varepsilon_1 + \cdots + \varepsilon_i = 0$, it must be the case that $\varepsilon_i > 0$. Consider the set of entries of T that are $\geqslant T_{2,i}$ except for the entry $T_{2,i}$ itself. This involves all entries in the first $i - 1$ columns, plus the $(1, i)$ entry, and possibly some entries in columns to the right. However the latter entries are zero in ε, so the corresponding sum in ε is ε_i. Hence $\varepsilon_i \leqslant 0$ and we have a contradiction.

The preceding argument proves that T is extremal and that the corresponding portion of the transportation polytope with content dominated by $\mathrm{co}(T)$ is 0-dimensional. The remaining conclusions now follow from Theorems 6.1 and 6.4. $\square$

We remark that the mere fact that the set of two-rowed plane partitions is closed under rescaling shows that extremality implies 0-dimensionality in the above context.

The following result shows that if the polytope $Q(\alpha, \beta; \gamma)$ includes a contingency table with content γ, and it is possible to stay in the polytope by moving infinitesimally in a direction with $0, \pm 1$ coordinates, then there must be another integer point in the polytope along this line.

LEMMA 6.10. *Let γ be a partition of length $\leqslant l$ and ε a vector of length l with $0, \pm 1$ coordinates having sum 0. If $\mathrm{co}(\gamma - t\varepsilon) \leqslant \gamma$ for some $t > 0$, then $\mathrm{co}(\gamma - \varepsilon) \leqslant \gamma$.*

Proof. By convexity considerations we may assume $t < 1/2$, so that $\gamma_i - t\varepsilon_i > \gamma_j - t\varepsilon_j$ whenever $\gamma_i > \gamma_j$. Moreover, we may permute the coordinates of ε if necessary so that if $\gamma_i = \gamma_j$ and $i < j$, then $\varepsilon_i \leqslant \varepsilon_j$. Thus $\gamma - t\varepsilon$ has weakly decreasing coordinates and the condition $\mathrm{co}(\gamma - t\varepsilon) \leqslant \gamma$ amounts to having $\varepsilon_1 + \cdots + \varepsilon_i \geqslant 0$ for all i. Therefore if we sum the coordinates of ε in positions where $\gamma_i > k$, or $\gamma_i = k$ and $\varepsilon_i = -1$, we obtain

$$(m_{k+1}^+ - m_k^-) + (m_{k+2}^+ - m_{k+1}^-) + \cdots \geqslant 0, \tag{6.3}$$

where $m_k^\pm$ denotes the number of indices i where $\gamma_i = k$ and $\varepsilon_i = \pm 1$.

On the other hand, given an integer vector δ of length l with the same sum as γ, let

$$\sigma_k(\delta) = \sum_i \max(\delta_i - k, 0) = \sum_{j>k} (j-k) m_j(\delta),$$

where $m_k(\delta)$ denotes the number of indices i where $\delta_i = k$. If δ is a partition of m, then $m - \sigma_k(\delta)$ is the sum of the k largest parts in the conjugate partition δ'. Recalling that conjugation is order-reversing with respect to dominance, it follows that $\mathrm{co}(\delta) \leqslant \gamma$ if and only if $\sigma_k(\delta) \leqslant \sigma_k(\gamma)$ for all $k \geqslant 0$. Moreover, this holds whether or not δ is a partition, since the quantities $\sigma_k(\cdot)$ are permutation invariant. Now consider that (6.3) implies

$$\begin{aligned}
\sigma_k(\gamma) - \sigma_k(\gamma - \varepsilon) &= \sum_{j>k} (j-k)(m_j(\gamma) - m_j(\gamma - \varepsilon)) \\
&= \sum_{j>k} (j-k)(m_j^+ + m_j^- - m_{j+1}^+ - m_{j-1}^-) = \sum_{j>k} (m_j^+ - m_{j-1}^-) \geqslant 0.
\end{aligned}$$

Hence γ dominates the content of $\gamma - \varepsilon$. $\square$

Proposition 6.11. *If $T \in C(\alpha,\beta)$ is a 3×3 contingency table with content γ, then the following are equivalent:*

(a) *T is extremal,*

(b) *T is a plane partition and $\mathrm{co}(T \pm E) \not\leqslant \gamma$, where*

$$E := \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

(c) *the polytope $Q(\alpha,\beta;\gamma)$ is 0-dimensional.*

Moreover, when these equivalent conditions hold, $\alpha\beta\gamma$ is stable and $g(\alpha\beta\gamma) = 1$.

Proof. The fact that (c) implies (a) is immediate. We also know that extremality of T forces T to be a plane partition (Theorem 6.4), and it is clear that if either $\mathrm{co}(T+E) \leqslant \gamma$ or $\mathrm{co}(T-E) \leqslant \gamma$ then T cannot be extremal. Thus (a) implies (b) and it remains to prove that (b) implies (c). (The final assertions are then consequences of Theorems 6.1 and 6.4.)

Henceforth we may assume that T is a plane partition.

Any point in the polytope $Q(\alpha,\beta;\gamma)$ may be represented in the form $T+\varepsilon$, where ε is a 3×3 real matrix with zero row and column sums such that $\mathrm{co}(T+\varepsilon) \leqslant \gamma$. Assuming temporarily that T has distinct entries, the constraint $\mathrm{co}(T+\varepsilon) \leqslant \gamma$ implies

$$\varepsilon_1 + \cdots + \varepsilon_i \leqslant 0 \quad (1 \leqslant i < 9),$$

where ε_i denotes the entry of ε in the same position as the i-th largest entry of T.

There are 42 ways to arrange the 9 entries of T, corresponding to the 42 standard Young tableaux of shape 333. We have checked by machine computation that the above constraints on ε in 36 of these cases, together with the row and column sum constraints, can only be satisfied when $\varepsilon = 0$. The exceptional cases correspond to the tableaux

$$\begin{matrix} 1 & 2 & 6 \\ 3 & 5 & 7 \\ 4 & 8 & 9 \end{matrix} \qquad \begin{matrix} 1 & 2 & 6 \\ 3 & 4 & 7 \\ 5 & 8 & 9 \end{matrix} \qquad \begin{matrix} 1 & 2 & 5 \\ 3 & 6 & 7 \\ 4 & 8 & 9 \end{matrix}$$

and their transposes. In each of these cases, it is straightforward to show that ε must be a scalar multiple of the matrix E.

If we now relax the hypothesis that the entries of T are distinct, there can only be *more* constraints on ε beyond those arising from a single Young tableau. We conclude that

$$Q(\alpha,\beta;\gamma) = \{T + tE : t \in [a,b]\}$$

for some closed interval $[a,b]$ containing 0. Thus if $Q(\alpha,\beta;\gamma)$ has positive dimension, there must be some nonzero t such that $\mathrm{co}(T+tE) \leqslant \gamma$. However, all entries of E are $0, \pm 1$, so this forces $\mathrm{co}(T+E) \leqslant \gamma$ or $\mathrm{co}(T-E) \leqslant \gamma$ by Lemma 6.10. Hence (b) implies (c). □

7. More polytopes and more stability

Given a triple of partitions $\alpha\beta\gamma$ of size m, let us define

$$f(\alpha\beta\gamma) = \text{multiplicity of } I_\gamma \text{ in } M_\alpha \otimes I_\beta = \dim(M_\alpha \otimes I_\beta \otimes I_\gamma)^{S_m}.$$

Our goal in this section is to show that known combinatorial formulas for these multiplicities are expressible as counts of integer points in certain explicit polytopes. This will allow us to identify more stable triples by techniques similar to those of the previous section.

Noting that I_β is a submodule of M_β, we have $g(\alpha\beta\gamma) \leqslant f(\alpha\beta\gamma) \leqslant h(\alpha\beta\gamma)$. Thus any triple that can be proved stable by establishing bounds on h-multiplicities also has bounded f-multiplicities. The catch is that the polytopes associated to f-multiplicities have more complicated descriptions and it is harder to identify when they are 0-dimensional.

Let $<_L$ denote a lexicographic total order of $(\mathbb{Z}^{>0})^2$, so that $(i,j) <_L (i',j')$ if $i < i'$ or $i = i'$ and $j < j'$. We say that a Young tableau T with entries chosen from $(\mathbb{Z}^{>0})^2$ is semistandard if its rows are weakly increasing and columns strictly increasing with respect to this order. The *bi-content* of T is the pair (α, β) consisting of the row-sum vector α and column-sum vector β of the matrix whose (i,j)-entry records the number of occurrences of (i,j) in the tableau T. The *reading word* of T is an ordering of the entries so that each (i,j) in row k precedes every (i',j') in row k' if $i < i'$, or $i = i'$ and $k < k'$, or $i = i'$, $k = k'$, and $j > j'$. We say that the reading word satisfies the *Yamanouchi condition* if $N(j,l) \geqslant N(j+1,l)$ for all j and l, where $N(j,l)$ denotes the number of terms of the form $(*,j)$ among the first l entries of the word.

The following result is well-known, although not typically expressed in this form.

Proposition 7.1. *The multiplicity $f(\alpha\beta\gamma)$ is the number of semistandard tableaux of shape γ and bi-content (α,β) whose reading word satisfies the Yamanouchi condition.*

Proof. If we assign the weight x_j to each entry (i,j), one sees that the generating function for all semistandard $(\mathbb{Z}^{>0})^2$-tableaux of shape γ and bi-content $(\alpha,*)$ is a sum of products skew Schur functions; namely,

$$\sum s_{\mu^{(1)}} s_{\mu^{(2)}/\mu^{(1)}} \cdots s_{\gamma/\mu^{(a-1)}}, \tag{7.1}$$

where the sum ranges over nested sequences of partitions $\mu^{(1)} \subset \mu^{(2)} \subset \cdots \subset \mu^{(a)} = \gamma$ such that $\mu^{(i)}$ has size $\alpha_1 + \cdots + \alpha_i$ and a is the length of α. By the Littlewood-Richardson Rule, it follows that the coefficient of the Schur function s_β in (7.1) is the number of such tableaux with bi-content (α,β) whose reading word satisfies the Yamanouchi condition. By Exercise 7.84.a in [**EC2**], it follows that this is $f(\alpha\beta\gamma)$. $\square$

Lemma 7.2. *If α, β, and γ have respective lengths a, b, c, then $f(\alpha\beta\gamma)$ is the number of integer points in the polytope defined by the constraints*

$$x(i,j,k) \geqslant 0 \quad (1 \leqslant i \leqslant a,\ 1 \leqslant j \leqslant b,\ 1 \leqslant k \leqslant c),$$

$$\sum_{j,k} x(i,j,k) = \alpha_i, \quad \sum_{i,k} x(i,j,k) = \beta_j, \quad \sum_{i,j} x(i,j,k) = \gamma_k, \tag{7.2}$$

$$\sum_{(i',j') <_L (i,j)} x(i',j',k) \geqslant \sum_{(i',j') \leqslant_L (i,j)} x(i',j',k+1), \tag{7.3}$$

$$\sum_{(i',k') <_L (i,k)} x(i',j,k') \geqslant \sum_{(i',k') \leqslant_L (i,k)} x(i',j+1,k'). \tag{7.4}$$

It should be emphasized that in (7.3) and (7.4), the indices i, j, k parameterize the inequalities; the indices i', j', k' are summation variables.

Note that interchanging β and γ is a permutation of the coordinates of this polytope.

Proof. Given a $(\mathbb{Z}^{>0})^2$-tableau T, let $x(i,j,k)$ be the number of times (i,j) appears in row k. Given that T has weakly increasing rows, it is clear that the coordinates $x(i,j,k)$ uniquely determine T. Moreover, the line sum constraints in (7.2) characterize those tableaux with bi-content (α, β) and shape γ. By Proposition 7.1, what remains is to show that (7.3) characterizes those tableaux that are semistandard (i.e., column increasing), and (7.4) characterizes those whose reading word satisfies the Yamanouchi condition.

If there is a column violation, then there is an entry (i,j) in row $k+1$ that is $\leqslant_L$ the entry above it in row k. Thus there are strictly more entries in row $k+1$ that are $\leqslant_L (i,j)$ than there are entries in row k that are $<_L (i,j)$ and (7.3) is violated. Conversely, if (7.3) is violated, then the rightmost occurrence of an entry $\leqslant_L (i,j)$ in row $k+1$ is below an element in row k that is $\geqslant_L (i,j)$ and T is not column-strict.

In the reading word of T, the $x(i,j,k)$ entries (i,j) in row k occur immediately after the $x(i,j+1,k)$ entries $(i,j+1)$. It follows that if there is a violation of the Yamanouchi condition, then there must be one in which the count of $(*,j)$'s is less than the count of $(*,j+1)$'s in the subword ending with all of the occurrences of $(i,j+1)$ in row k, for some i, j and k. This is exactly the condition forbidden by (7.4). $\square$

Remark 7.3. If we drop the Yamanouchi condition in Proposition 7.1, we obtain a description of the monomial expansion of (7.1). Bearing in mind the duality between monomials and the permutation modules M_β, it follows that $h(\alpha\beta\gamma)$ is the number of semistandard tableaux of shape γ and bi-content (α, β), or equivalently, the number of integer points in the polytope obtained by omitting the constraints in (7.4) from the description in Lemma 7.2. In this context, Conjecture 6.7 amounts to the assertion that this polytope is 0-dimensional whenever it has only one integer point (cf. Remark 5.5).

THEOREM 7.4. *If $f(n \cdot \alpha\beta\gamma) = 1$ for all $n \geqslant 1$ (or equivalently, if the polytope defined in Lemma 7.2 is 0-dimensional and consists of a single integer vertex), then*

(a) *there is a unique partition $\alpha^+ \geqslant \alpha$ such that $g(\alpha^+\beta\gamma) > 0$,*
(b) *the triple $\alpha^+\beta\gamma$ is stable, and*
(c) *we have $f(n \cdot \alpha^+\beta\gamma) = 1$ for all $n \geqslant 1$.*

Note that $f(n \cdot \alpha\beta\gamma) \geqslant g(n \cdot \alpha\beta\gamma) \geqslant 1$ if $g(\alpha\beta\gamma) > 0$, hence

COROLLARY 7.5. *If $g(\alpha\beta\gamma) > 0$ and $f(n \cdot \alpha\beta\gamma) \leqslant 1$ for all $n \geqslant 1$, then $\alpha\beta\gamma$ is stable.*

Proof of Theorem 7.4. Given that the stated condition holds, there must be exactly one irreducible summand I_{α^+} of M_α such that $g(\alpha^+\beta\gamma) > 0$ (and $g(\alpha^+\beta\gamma) = 1$ in that case) or we contradict the fact that $f(\alpha\beta\gamma) = 1$. Since all such summands satisfy $\alpha^+ \geqslant \alpha$, we obtain (a). For (b), note that $I_{\lambda+n\alpha^+}$ is a summand of $M_{\lambda+n\alpha}$, hence

$$g(\lambda\mu\nu + n \cdot \alpha^+\beta\gamma) \leqslant f(\lambda\mu\nu + n \cdot \alpha\beta\gamma)$$

for all triples $\lambda\mu\nu$. However, the polytope described in Lemma 7.2 fits the form of (4.2), so Proposition 4.4 and our hypothesis that $f(n \cdot \alpha\beta\gamma) = 1$ for all n implies that $f(\lambda\mu\nu + n \cdot \alpha\beta\gamma)$ is bounded for all triples $\lambda\mu\nu$ and thus $\alpha^+\beta\gamma$ is stable. Finally, recall by Proposition 5.1(a) that $M_{n\alpha^+}$ embeds in $M_{n\alpha}$, hence

$$1 = f(n \cdot \alpha\beta\gamma) \geqslant f(n \cdot \alpha^+\beta\gamma) \geqslant g(n \cdot \alpha^+\beta\gamma) = 1$$

by Proposition 4.2. Thus (c) holds. □

PROPOSITION 7.6. *If γ is a partition of m obtained by moving one cell in the diagram of β to* (a) *the first row or* (b) *the first column, then the triple $((m-1,1),\beta,\gamma)$ is stable.*

Proof. It is well-known that $g((m-1,1),\beta,\gamma) = 1$ in this case (e.g., see Exercise 7.81 in [**EC2**]), so it suffices by Corollary 7.5 to show that $f((nm-n,n),n\beta,n\gamma) \leqslant 1$. In view of Proposition 7.1, we thus seek to show that there is at most one semistandard tableau T of shape $n\gamma$ and bi-content $((nm-n,n),n\beta)$ whose reading word satisfies the Yamanouchi condition. A general feature of such tableaux is that all of the entries of the form $(1,k)$ must appear in the leftmost columns of row k, and thus the remaining n entries of the form $(2,*)$ are forced to appear along the outer boundary of T.

In case (a), the first row of γ is strictly longer than that of β, so the $(2,*)$-entries must necessarily appear at the right end of the first row. Otherwise, there would have to be more than $n\beta_1$ occurrences of $(1,1)$ in the first row of T, a contradiction. Thus the forced occurrences of $(1,k)$ leave available for placement only n copies of $(2,i)$, assuming i is the row number where γ is shorter than β. As we have noted, these entries must appear at the end of the first row, so there is at most one such T.

Similarly in case (b), we claim that the n entries of the form $(2,*)$ must be evenly distributed at the bottom of the first n columns of T. Otherwise, the first column of T

would have to be filled with the entries $(1,k)$ for $1 \leqslant k \leqslant \ell(\gamma)$. Since γ has a longer first column than β, this contradicts the fact that T has bi-content $((nm-n,n),\beta)$. Accounting for the forced placement of the entries of the form $(1,k)$, this means that the entries at the bottom of the first n columns of T must all be $(2,i)$, where i is the row where γ is shorter than β. Thus again, T is uniquely determined. $\square$

We remark that Theorem 6.1 often cannot be used to directly prove the stability of the Kronecker triples identified in the above proposition. Examples of this include $(31,22,211)$, $(41,32,311)$, $(41,311,221)$, and $(41,221,2111)$.

8. The stability of $(22,22,22)$

The main goal of this section is to prove that the triple $(22,22,22)$ is stable. This particular triple is noteworthy since it is the only triple of size $\leqslant 5$ whose stability is not a direct consequence of either Theorems 6.1 or 7.4.

The following result reduces the stability of this triple to a question about Kronecker coefficients involving two-rowed partitions. It also provides a very simple alternative proof of the stability of any triple of the form (α,α,m).

Proposition 8.1. *Given a triple $\alpha\beta\gamma$, the sequences $g(\lambda\mu\nu + n\cdot\alpha\beta\gamma)$ are bounded as a function of n for all triples $\lambda\mu\nu$ if they are bounded for all triples such that $\ell(\lambda) \leqslant \ell(\alpha)$, $\ell(\mu) \leqslant \ell(\beta)$, and ν is a multiple of γ.*

Proof. Given an S_m-module M, let $\mathrm{Ind}^r M$ and $\mathrm{Res}^r M$ denote the induction and restriction of M to S_{m+r} and S_{m-r}, respectively. By Young's Rule, one knows that $\mathrm{Ind}^1 I_\gamma$ and $\mathrm{Res}^1 I_\gamma$ are the direct sums of all irreducible S_{m+1}-modules (resp., S_{m-1}-modules) I_δ such that δ is obtained by increasing (resp., decreasing) the length of one row of γ by one in all possible ways. Thus if ν is a partition of r, the induced module $\mathrm{Ind}^r I_{n\gamma}$ will necessarily contain at least one copy of $I_{\nu+n\gamma}$ (and typically many such copies). If $\alpha\beta\gamma$ is a triple of size m, we therefore have

$$\begin{aligned} g(\lambda\mu\nu + n\cdot\alpha\beta\gamma) &\leqslant \dim\left(I_{\lambda+n\alpha}\otimes I_{\mu+n\beta}\otimes \mathrm{Ind}^r I_{n\gamma}\right)^{S_{nm+r}} \\ &= \dim\left(\mathrm{Res}^r I_{\lambda+n\alpha}\,\otimes\,\mathrm{Res}^r I_{\mu+n\beta}\,\otimes\, I_{n\gamma}\right)^{S_{nm}} \end{aligned}$$

by Frobenius Reciprocity.

The summands in the restriction of $I_{\lambda+n\alpha}$ involve sequences of r decrements of the parts of $\lambda+n\alpha$. It follows that there is a fixed multiset of integer vectors θ (independent of n but depending on λ and α), each summing to 0 and of length $\leqslant \max(\ell(\alpha),\ell(\lambda))$, such that $I_{\theta+n\alpha}$ is a summand of the restriction if and only if $\theta+n\alpha$ is a partition (and hence this summand occurs for all n sufficiently large). In the same way, one may write $\mathrm{Res}^r I_{\mu+n\beta}$ as a similar fixed sum of terms $I_{\phi+n\beta}$ for n sufficiently large. Thus it suffices to separately bound the Kronecker coefficients

$$g(\theta+n\alpha,\phi+n\beta,n\gamma)$$

for each θ and ϕ arising. If the minimum n such that $\theta + n\alpha$ and $\phi + n\beta$ are partitions occurs at $n = n_0$, then we have in effect replaced $\lambda\mu\nu$ with $(\theta + n_0\alpha, \phi + n_0\beta, n_0\gamma)$.

Beyond the fact that we have replaced ν with a multiple of γ, the key point of this reduction is that while the sizes of λ and μ may have grown, the extent that their lengths exceed the lengths of α and β have not. Thus by symmetry considerations, if we now repeat the argument, but with the roles of $\mu + n\beta$ and $\nu + n\gamma$ switched, we will lose the feature that ν is a multiple of γ, but its length will remain bounded by $\ell(\gamma)$. Applying this switch a second time, we can assume all of the lengths of λ, μ and ν are bounded by the lengths of α, β and γ, and in addition, assume either that λ is a multiple of α, or μ a multiple of β, or ν a multiple of γ. $\square$

LEMMA 8.2. *If $\lambda = (n, n)$ and α is a partition of length $\leqslant 4$ and size $2n$, then the Kostka number $K_{\lambda,\alpha}$ is nonzero only if $\alpha_1 \leqslant n$. In that case, we have*

$$K_{\lambda,\alpha} = 1 + \min(\alpha_4, n - \alpha_1).$$

Proof. It is well-known that $K_{\lambda,\alpha}$ is invariant under permutations of the parts of α, so $K_{\lambda,\alpha}$ is the number of semistandard tableaux of shape λ and content $(\alpha_1, \alpha_3, \alpha_4, \alpha_2)$. In any such tableaux, all entries equal to 1 necessarily occur in the first row (hence the condition $\alpha_1 \leqslant n$) and all entries equal to 4 necessarily occur in the rightmost columns of the second row. Moreover, every column has either a 1 or a 4 (or both). Otherwise, we would have $n > \alpha_1 + \alpha_2 \geqslant \alpha_3 + \alpha_4$, contradicting the fact that α is a partition of $2n$. The remaining entries in the two rows thus have no common columns.

Such tableaux are uniquely determined by (say) how many 2's occur in row 1. If there are i such 2's, then $n - i - \alpha_1$ entries in row 1 are 3's, $\alpha_3 - i$ entries in row 2 are 2's, and $\alpha_1 + \alpha_4 - n + i$ entries in row 2 are 3's. Given that these amounts are nonnegative; i.e.,

$$\max(0, n - \alpha_1 - \alpha_4) \leqslant i \leqslant \min(\alpha_3, n - \alpha_1),$$

then such a tableau exists. The number of choices for i is therefore

$$1 + \min(\alpha_3, n - \alpha_1, \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 - n).$$

Noting that $\alpha_1 + \alpha_3 + \alpha_4 - n = n - \alpha_2$, this agrees with the claimed formula. $\square$

In the following, $\lfloor x \rfloor$ denotes the greatest integer $\leqslant x$.

LEMMA 8.3. *If $n \geqslant 1$, $k \geqslant 0$, $\mu = (n + 2k + 1, n - 1)$ and $\nu = (n + k, n + k)$, then*

(a) *$h(\nu\nu\nu) = \lfloor (n + k + 2)^2/4 \rfloor$,*
(b) *$h(\mu\nu\nu) = \lfloor (n + 1)^2/4 \rfloor$, and*
(c) *if $n \geqslant k - 1$, then $h(\mu\mu\nu) = \lfloor (n - k)^2/4 \rfloor$.*

Proof. In order to use (6.1) to evaluate these multiplicities, we need to identify the contingency tables of types (ν,ν), (μ,ν), and (μ,μ) (respectively), and evaluate the corresponding Kostka numbers. It is not hard to see that the relevant tables are

$$\begin{bmatrix} n+k-i & i \\ i & n+k-i \end{bmatrix}, \quad \begin{bmatrix} n+k-i & k+i+1 \\ i & n-i-1 \end{bmatrix}, \quad \begin{bmatrix} n+2k-i+1 & i \\ i & n-i-1 \end{bmatrix},$$

where (respectively) $0 \leqslant i \leqslant n+k$, $0 \leqslant i \leqslant n-1$, and $0 \leqslant i \leqslant n-1$. In the last case, we should further restrict i to the range $k+1 \leqslant i \leqslant n-1$; otherwise, the largest entry of the table will exceed $\nu_1 = n+k$ and the corresponding Kostka number will be 0. In particular, there will be no relevant tables in this last case unless $n \geqslant k+2$. However, since the claimed formula in (c) evaluates to 0 when $n-k \in \{0, \pm 1\}$, there is no harm in assuming $n \geqslant k+2$ henceforth for this case.

Using Lemma 8.2 to evaluate the Kostka numbers arising in these cases, we see that

$$\begin{aligned} K_{\nu,(n+k-i,n+k-i,i,i)} &= 1 + \min(i, n+k-i) = \min(i+1, n+k+1-i), \\ K_{\nu,(n+k-i,k+i+1,n-i-1,i)} &= 1 + \min(i, n-1-i) = \min(i+1, n-i), \\ K_{\nu,(n+2k-i+1,n-i-1,i,i)} &= 1 + \min(i-k-1, n-1-i) = \min(i'+1, n-k-1-i'), \end{aligned}$$

where $i' = i-k-1$. (Note also that $0 \leqslant i' \leqslant n-k-2$.) Thus (6.1) implies

$$h(\mu\nu\nu) = \sum_{i=0}^{n-1} \min(i+1, n-i) = \lfloor (n+1)^2/4 \rfloor,$$

proving (b). The analogous sums for $h(\nu\nu\nu)$ and $h(\mu\mu\nu)$ are termwise the same as one would obtain after replacing n with $n+k+1$ or $n-k-1$ in the above sum. □

Proposition 8.4. *The triple* $(22, 22, 22)$ *is stable.*

Proof. After confirming that $g(22, 22, 22) > 0$, it suffices to show that the sequences $g(\lambda\mu\nu + n \cdot (11, 11, 11))$ are bounded for all triples $\lambda\mu\nu$. By Proposition 8.1, this follows if we can show that $g((n+2k-a, n+a), (n+2k-b, n+b), (n+k, n+k))$ is a bounded function of n for integers $k \geqslant a, b \geqslant 0$. Now set $\mu = (n+2k+1, n-1)$ and $\nu = (n+k, n+k)$. By comparing irreducible multiplicities, it is easy to check that

$$M_\nu \cong I_{(n+k,n+k)} \oplus I_{(n+k+1,n+k-1)} \oplus \cdots \oplus I_{(n+2k,n)} \oplus M_\mu.$$

Thus for $n \geqslant k-1$, Lemma 8.3 implies

$$\begin{aligned} \sum_{a,b} g((n+2k-a, &n+a), (n+2k-b, n+b), \nu) = h(\nu\nu\nu) - 2h(\mu\nu\nu) + h(\mu\mu\nu) \\ &= \frac{1}{4}(n+k+2)^2 - \frac{1}{2}(n+1)^2 + \frac{1}{4}(n-k)^2 + \varepsilon = \frac{1}{2}(k+1)^2 + \varepsilon, \end{aligned}$$

where ε is an error term that depends only on the parities of n and k. A careful accounting for this error term shows that in fact $\varepsilon = (-1)^n/2$ if k is even and $\varepsilon = 0$ if k is odd. In any case, it is clear that the above sum is bounded for fixed k. □

REMARK 8.5. (a) In the special case $k = 0$, the above calculation shows that

$$g(n^2, n^2, n^2) = \frac{1}{2} + \varepsilon = \frac{1}{2}(1 + (-1)^n).$$

(b) There are known combinatorial formulas for $g(\lambda\mu\nu)$ when (say) λ and μ have at most two rows, but they are rather complicated. See pp. 124–128 of [**BMS**] for the first such formula that is manifestly positive. On the other hand, the inequality

$$g((n + 2k - a, n + a), (n + 2k - b, n + b), (n + k, n + k)) \leqslant \lfloor (k - a + 2)/2 \rfloor$$

is an easy consequence of a (positive) formula due to Brown, van Willigenburg and Zabrocki (see Corollary 3.2 of [**BvZ**]), so this provides a more expedient proof of Proposition 8.4 that bypasses the need for Lemmas 8.2 and 8.3.

9. Reducibility and multivariate stability

A triple $\alpha\beta\gamma$ in the Kronecker semigroup $\mathcal{G}$ (or more generally in any additive semigroup) is said to be *reducible* if it has a nontrivial representation of the form

$$\alpha\beta\gamma = \lambda\mu\nu + \rho\theta\tau, \quad \lambda\mu\nu, \rho\theta\tau \in \mathcal{G}.$$

Otherwise, it is irreducible.

Note that if a reducible triple is stable, then each of its summands must also be stable. Indeed, by monotonicity we have

$$g(\rho\theta\tau + n \cdot (\alpha\beta\gamma + \lambda\mu\nu)) \;\geqslant\; g(\rho\theta\tau + n \cdot \alpha\beta\gamma),$$

so if the sequence on the left is bounded, the same is true for the sequence on the right. An even stronger consequence of stability for reducible triples is that they imply boundedness and convergence of Kronecker coefficients in higher dimensional affine cones, not just along affine rays. For example, monotonicity implies

$$g(\rho\theta\tau + a \cdot (\alpha\beta\gamma + \lambda\mu\nu)) \leqslant g(\rho\theta\tau + m \cdot \alpha\beta\gamma + n \cdot \lambda\mu\nu) \leqslant g(\rho\theta\tau + b \cdot (\alpha\beta\gamma + \lambda\mu\nu))$$

for $a \leqslant m, n \leqslant b$, so we have the following.

PROPOSITION 9.1. *If $\alpha\beta\gamma, \lambda\mu\nu \in \mathcal{G}$ and $\alpha\beta\gamma + \lambda\mu\nu$ is stable, then the quantity*

$$g(\rho\theta\tau + m \cdot \alpha\beta\gamma + n \cdot \lambda\mu\nu)$$

is independent of m and n provided that both are sufficiently large.

For example, this applies to the stable triple $(31, 31, 22) = (2, 11, 11) + (11, 2, 11)$.

Example 9.2. A more extreme instance of multivariate stability may be constructed from the triples $(k, 1^k, 1^k)$ for $k \geqslant 1$. These triples are stable (Example 6.3(a)) and it is easy to see that they are irreducible. Furthermore, arbitrary sums of these triples have the form (m, α, α) for partitions α. Since these sums are again stable (Example 6.3(a)), it follows for example that

$$g(\lambda\mu\nu + n_1 \cdot (1,1,1) + n_2(2,11,11) + n_3(3,1^3,1^3) + n_4(4,1^4,1^4))$$

is independent of n_1, n_2, n_3, n_4 provided that all n_i are sufficiently large.

In Table 9.3 we list the (unordered) Kronecker triples of size $\leqslant 5$, with the stable and unstable cases segregated. The reducible triples are annotated with an 'r', and the stable triples are annotated with an indication of the source of a proof of stability: 'a' through 'e' refer to Examples 6.3(a) and 6.3(b), and Propositions 6.9, 7.6, and 8.4, respectively. For most unstable $\alpha\beta\gamma$ in this table, the least n such that $g(n \cdot \alpha\beta\gamma) > 1$ is $n = 2$. In all other cases of size $\leqslant 5$, this occurs at $n = 1$ and the annotation '1' is provided.

stable triples				unstable triples			
$(1,1,1)$	a						
$(2,2,2)$	a,r	$(2,11,11)$	a				
$(3,3,3)$	a,r	$(3,1^3,1^3)$	a	$(21,21,21)$			
$(3,21,21)$	a,r	$(21,21,1^3)$	b				
$(4,4,4)$	a,r	$(31,31,211)$	c,r	$(31,31,31)$	r	$(22,211,211)$	
$(4,31,31)$	a,r	$(31,22,211)$	d	$(31,211,211)$		$(211,211,211)$	
$(4,22,22)$	a,r	$(31,211,1^4)$	b				
$(4,211,211)$	a,r	$(22,22,22)$	e				
$(4,1^4,1^4)$	a	$(22,22,1^4)$	b				
$(31,31,22)$	d,r						
$(5,5,5)$	a,r	$(41,32,311)$	d,r	$(41,41,41)$	r	$(32,221,221)$	
$(5,41,41)$	a,r	$(41,32,221)$	c,r	$(41,32,32)$	r	$(32,221,21^3)$	
$(5,32,32)$	a,r	$(41,311,221)$	d,r	$(41,311,311)$	r	$(32,21^3,21^3)$	
$(5,311,311)$	a,r	$(41,311,21^3)$	c,r	$(41,221,221)$		$(311,311,311)$	r
$(5,221,221)$	a,r	$(41,221,21^3)$	d	$(41,21^3,21^3)$		$(311,311,221)$	1
$(5,21^3,21^3)$	a,r	$(41,21^3,1^5)$	b	$(32,32,32)$	r	$(311,311,21^3)$	
$(5,1^5,1^5)$	a	$(32,32,21^3)$	c,r	$(32,32,311)$	r	$(311,221,221)$	
$(41,41,32)$	d,r	$(32,221,1^5)$	b	$(32,32,221)$		$(311,221,21^3)$	
$(41,41,311)$	c,r	$(311,311,1^5)$	b	$(32,311,311)$	1,r	$(311,21^3,21^3)$	
				$(32,311,221)$		$(221,221,221)$	
				$(32,311,21^3)$		$(221,221,21^3)$	

Table 9.3: Stability and reducibility of Kronecker triples of size $\leqslant 5$.

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