Notes on Perron-Frobenius Theory

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1. Main results

The Perron-Frobenius Theorem is a collection of facts about the eigenvalues and eigenvectors of real nonnegative matrices. In these notes we provide complete proofs of the main results; the one non-trivial thing we take for granted is the existence of Jordan Canonical Form over the complex field.

We make no claims that anything stated here is new or original.

For a square complex matrix $A$, let $\rho(A)$ denote its spectral radius; i.e., the maximum of $|\lambda|$ as $\lambda$ varies over the eigenvalues of $A$. If $\rho(A) = |\lambda|$, we say that $\lambda$ is extremal.

**Lemma 1.** Assume $\rho = \rho(A) > 0$, and let $r$ denote the size of the largest Jordan block associated to an extremal eigenvalue of $A$.

(a) The rate of growth of the entries of $A^n$ is limited; namely,

$$(A^n)_{i,j} = O(n^{r-1}\rho^n) \quad \text{as } n \to \infty, \text{ for all } i, j.$$ 

(b) If $\lambda$ is the only extremal eigenvalue of $A$, then for all vectors $v$, the limit

$$L(v) := \lim_{n \to \infty} \frac{A^n v}{n^{r-1}\lambda^n}$$

converges, and the range of $L$ is a nonzero subspace of the $\lambda$-eigenspace of $A$.

**Proof.** Both claims are stable under changes of basis, so we may assume that $A$ is in Jordan Canonical Form. In addition, if the claims hold for $A_1$ and $A_2$, then they hold for their direct sum. Thus we may assume that $A$ consists of a single $r \times r$ Jordan block; i.e.,

$$A = \lambda + E,$$

where $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (if $r = 3$, say).
Since $E^r = 0$, one sees that
\[ A^n = (\lambda + E)^n = \lambda^n + \binom{n}{1} \lambda^{n-1} E + \cdots + \binom{n}{r-1} \lambda^{n-r+1} E^{r-1}, \]
and now (a) follows easily. For (b), we have
\[ \lim_{n \to \infty} \frac{A^n}{n^{r-1} \lambda^n} = \frac{1}{(r-1)!} \lambda^{r-1} E^{r-1}, \]
and the range of this operator is the coordinate space $\mathbb{C}e_1$, the $\lambda$-eigenspace of $A$. □

**Theorem 2.** If all sufficiently high powers of $A$ are real and positive, then the extremal eigenvalues of $A$ are real and positive (i.e., if $\lambda$ is extremal, then $\lambda = \rho(A) > 0$).

For a stronger version of this result, see Theorem 5 below.

We note that the above hypothesis does not require $A$ to be nonnegative. For example, if $A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$, then $A^n$ is positive for $n = 2$ and $n \geq 4$. (See also Corollary 10.)

**Proof.** (Shamelessly adapted from Wikipedia.) Let $\rho = \rho(A)$. We cannot have $\rho = 0$, otherwise $A$ would be nilpotent and all sufficiently high powers of $A$ would be 0.

Replacing $A$ with $A/\rho$, we may assume $\rho = 1$.

Now let $\lambda$ be an extremal eigenvalue of $A$. If $\lambda \neq 1$, then we can choose $m$ so that $A^m$ is positive and $\lambda^m$ has negative real part. If $a$ is the smallest diagonal entry of $A^m$, then $B = A^m - a/2$ is positive and has an eigenvalue $\mu = \lambda^m - a/2$ such that $|\mu| > 1$. However, $B \preceq A^m$ entry-wise, so the matrix entries of $B^n$ are bounded by the matrix entries of $A^{mn}$. Since $\rho(A) = 1$, the latter are of polynomial growth by Lemma 1. On the other hand, there is an eigenvector $v$ for $B$ such that $B^n v = \mu^n v$ has exponential growth, a contradiction. □

**Lemma 3.** If $A$ is (real and) nonnegative, then $A$ has a nonnegative eigenvector with eigenvalue $\rho(A)$.

**Proof.** We know that $\rho(A)$ is an eigenvalue of $A$ if $A$ is positive (Theorem 2), so the same is true for nonnegative $A$ by continuity.

For the eigenvectors, we may replace $A$ with $1 + A$. Indeed, this has no effect on the eigenspaces, but shifts the spectrum so that $\rho = \rho(A)$ is the unique extremal eigenvalue, and is positive. By Lemma 1(b), it follows that there is an integer $r \geq 1$ so that
\[ v \mapsto L(v) = \lim_{n \to \infty} \frac{A^n v}{n^{r-1} \rho^n} \]
is a nonzero linear map into the $\rho$-eigenspace of $A$. Thus there must be a coordinate vector $e_i$ such that $L(e_i)$ is a $\rho$-eigenvector (i.e., nonzero), and it is nonnegative, since $A$ is nonnegative. □
Recall that a directed graph is strongly connected if there is a directed path between every pair of vertices, or equivalently, there is a closed (directed) path that passes through every vertex. In particular, a graph with one vertex and no edges is strongly connected.

The support graph of a square matrix \( A = [a_{ij}] \) is a directed graph in which there is an edge from \( i \) to \( j \) if \( a_{ij} \neq 0 \). We will say that \( A \) is strongly connected if this associated graph is strongly connected. It is not hard to show that this is equivalent to the non-existence of a simultaneous permutation of the rows and columns of \( A \) having the block triangular form 
\[
\begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix},
\]
where \( A_{11} \) and \( A_{22} \) are square submatrices.

Note that the \( 1 \times 1 \) matrix \([0]\) is strongly connected; it is the only strongly connected matrix with spectral radius 0.

**Theorem 4.** If \( A \) is nonnegative and strongly connected, then

(a) \( A \) has a positive eigenvector \( v \) with eigenvalue \( \rho(A) \),
(b) the \( \rho(A) \)-eigenspace is one-dimensional,
(c) \( v \) is the unique nonnegative eigenvector of \( A \) (up to scalar multiples), and
(d) \( \rho(A) \) is a simple root of the characteristic polynomial of \( A \).

**Proof.** We may replace \( A \) by \( A + 1 \) so that by Lemma 3, \( \rho = \rho(A) \) is the only extremal eigenvalue of \( A \), and \( \rho > 0 \).

(a) Let \( v \) be a nonnegative eigenvector of \( A \) with eigenvalue \( \rho \), as provided by Lemma 3. If the first \( a \) coordinates of \( v \) are positive and the remaining \( b \) coordinates of \( v \) are zero, then the condition \( Av = \rho v \) forces the southwest \( b \times a \) submatrix of \( A \) to be 0, and thus \( A \) could not be strongly connected. Hence \( v \) must be positive.

(b) Now suppose that \( u \) is another eigenvector in the \( \rho \)-eigenspace. Replace \( u \) with \(-u\) if necessary so that at least one coordinate of \( u \) is positive. It follows that if \( c \) is the largest scalar such that \( v - cu \) is nonnegative, then \( v - cu \) has at least one zero coordinate, and hence cannot be an eigenvector, by the argument in the previous paragraph. Since \( v - cu \) belongs to the \( \rho \)-eigenspace, the only other possibility is that \( v = cu \).

(c) Let \( w^T \) be a positive left eigenvector\(^1\) for \( A \) with eigenvalue \( \rho \). (Apply (a) to \( A^T \).) If \( u \) is a nonnegative right eigenvector with eigenvalue \( \lambda \), we can evaluate \( w^T Av \) in two ways, obtaining \( \rho w^T u = w^T Au = \lambda w^T u \). However, \( w^T u \) is necessarily positive, so \( \lambda = \rho \) and \( u \) is a multiple of \( v \) by (b).

(d) Let \( r \) be the multiplicity of \( \rho \) as a root of the characteristic polynomial of \( A \). Since the \( \rho \)-eigenspace is one-dimensional, it must be the case that \( A \) has only one Jordan block associated to \( \rho \), and it has order \( r \). Thus by Lemma 1(b), some matrix entries of \( A^n \) grow at the asymptotic rate of \( n^{r-1} \rho^n \), so the same must be true for the coordinates of \( A^n u \) for any positive vector \( u \). However, for the positive eigenvector \( v \) we have \( A^n v = \rho^n v \), a contradiction unless \( r = 1 \). \( \Box \)

\(^1\)It would also be sensible to call \( w^T \) a right eigenvector, since \( w^T \mapsto w^T A \) is a right action for \( A \). This is one of those situations where standard terminology is in conflict, or at least dyslexic.
The following result strengthens Theorem 2.

**Theorem 5.** If all sufficiently high powers of $A$ are real and positive, then all of the conclusions of Theorem 4 hold, and we have

$$A^n = \frac{1}{w^Tv} \rho^n vw^T + o(\rho^n) \text{ as } n \to \infty,$$

where $w^T$ and $v$ denote left and right eigenvectors for $A$ with eigenvalue $\rho = \rho(A)$.

**Proof.** Choose $m$ so that $A^m$ is positive. We know that $\rho$ is the only extremal eigenvalue of $A$ (Theorem 2), and any extremal eigenvalue/vector for $A$ is also extremal for $A^m$, so Theorem 4 (applied to $A^m$) implies that $\rho$ has multiplicity 1 as an eigenvalue of $A$, and the corresponding eigenvector is positive (if suitably normalized). Similarly, there is no other nonnegative eigenvector for $A$ since the same is true for $A^m$.

To prove the asymptotic formula, note that both sides are invariant under changes of basis and choices of eigenvectors, so we may assume that $A$ is in Jordan Canonical Form. Since $\rho$ has multiplicity 1, there is a $1 \times 1$ Jordan block with eigenvalue $\rho$, and all other blocks have eigenvalues $\mu$ with $|\mu| < \rho$. Powers of these blocks therefore grow at rates asymptotically slower than $\rho^n$ (Lemma 1), so if the blocks are ordered so that $e_1^T$ and $e_1$ are left and right $\rho$-eigenvectors, then $A^n = \text{diag}(\rho^n, 0, \ldots, 0) + o(\rho^n) = \rho^n e_1 e_1^T + o(\rho^n)$. □

If $A$ is merely nonnegative and strongly connected, then it may happen that every power of $A$ has entries that vanish, and there may be extremal eigenvalues in addition to $\rho(A)$. In fact, these two (related) misfeatures are the subject of the next section.

2. Periodicity

Define a directed graph to be $m$-cyclic if the vertices may be partitioned into disjoint (nonempty) blocks $V_k$ ($0 \leq k < m$) so that every edge is directed from a vertex in $V_k$ to a vertex in $V_{k+1}$ (subscripts taken mod $m$). In the case $m = 2$, this is the same as the graph being bipartite, but being $m$-cyclic and $m$-partite are not the same for $m > 2$.

Similarly we will say that a (square) matrix $A$ is $m$-cyclic if the support graph of $A$ is $m$-cyclic. Illustrating this in the case $m = 3$, this amounts to saying that there is a suitable simultaneous permutation of the rows and columns of $A$ that has the block form

$$
\begin{bmatrix}
0 & A_{12} & 0 \\
0 & 0 & A_{23} \\
A_{31} & 0 & 0
\end{bmatrix}.
$$

Note that the diagonal blocks of zeroes must be square.

**Lemma 6.** If $\mu_1, \ldots, \mu_n$ are the nonzero eigenvalues (with multiplicity) of the product $A_{12}A_{23}\cdots A_{m1}$ of the blocks of an $m$-cyclic matrix $A$, then $A$ has exactly $mn$ nonzero eigenvalues (counted by multiplicity), and they are precisely the $m$-th roots of $\mu_1, \ldots, \mu_n$.
Proof. Let $B = A_{12}A_{23} \cdots A_{m1}$. We are given that $\det(1 - tB) = (1 - \mu_1 t) \cdots (1 - \mu_n t)$. Taking the determinant of the identity

$$
\begin{bmatrix}
1 & -tA_{12} & 0 \\
0 & 1 & -tA_{23} \\
-tA_{31} & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 \\
t^2 A_{23} A_{31} & 1 & 0 \\
t A_{31} & 0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
1 - t^3 A_{12} A_{23} A_{31} & -tA_{12} & 0 \\
0 & 1 & -tA_{23} \\
0 & 0 & 1
\end{bmatrix}
$$

reveals that (in the case $m = 3$)

$$
\det(1 - tA) = \det(1 - t^3 B) = (1 - \mu_1 t^3) \cdots (1 - \mu_n t^3),
$$

and the argument for general $m$ is essentially the same. □

By symmetry, the matrix factors appearing in $B = A_{12}A_{23} \cdots A_{m1}$ can be permuted cyclically without affecting the result, yielding the following amusement.

**Corollary 7.** If $A_1, \ldots, A_m$ are rectangular matrices, then the nonzero eigenvalues of $A_1 \cdots A_m$ (with multiplicity) are unchanged by cyclic permutations of the factors, given that all $m$ permuted products are defined (and therefore are square).

Define the periodicity of a square matrix $A$ to be the greatest common divisor of the lengths of all closed directed paths in its support graph. Note that the periodicity of the $1 \times 1$ matrix $[0]$ is $\infty$; for all other strongly connected matrices it is an integer $p \geq 1$.

**Theorem 8.** Let $\omega$ be a primitive $p$-th root of unity. If $A$ is nonnegative and strongly connected with periodicity $p < \infty$, then

(a) $A$ is $p$-cyclic,
(b) the eigenvalues of $A$ (with multiplicity) are stable under multiplication by $\omega$,
(c) the extremal eigenvalues of $A$ are $\omega^k \rho$ ($0 \leq k < p$), and
(d) each extremal eigenvalue is a simple root of the characteristic polynomial of $A$.

Proof. (a) Suppose that there are directed paths of lengths $k$ and $l$ from vertex 1 to vertex $i$ in the support graph of $A$. There must also be a directed path, say of length $h$, that returns from vertex $i$ to vertex 1. Thus there are closed paths of lengths $k + h$ and $l + h$, so both lengths must be divisible by $p$ and $k = l \mod p$. In other words, all paths from vertex 1 to vertex $i$ have the same length $\mod p$.

We may therefore partition the vertices of the support graph into $p$ blocks so that block $k$ consists of all vertices reachable from vertex 1 by a directed path of length $k \mod p$ ($0 \leq k < p$). If there were an edge directed from block $k$ to block $l$, then there would be a directed path of length $k + 1 \mod p$ from vertex 1 to a vertex in block $l$, so this could happen only if $l = k + 1 \mod p$. Thus $A$ is $p$-cyclic.

Having proved that $A$ is $p$-cyclic, (b) follows immediately from Lemma 6.

It also follows that $A^p$ is block-diagonal, say $A^p = \text{diag}(A_0, \ldots, A_{p-1})$. 

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Given a set of positive integers with greatest common divisor \( p \), one knows by a theorem of Schur\(^2\) that every sufficiently large multiple of \( p \) is a nonnegative integer combination of those integers. Thus by extending a single closed path that passes through every vertex, one can find such paths having a length equal to any sufficiently large multiple of \( p \). Since there must be a directed path of length divisible by \( p \) between any two vertices in the same block, it follows that there must also be directed paths whose lengths are any sufficiently large multiple of \( p \). That is, all sufficiently high powers of \( A_0, \ldots, A_{p-1} \) are positive.

By Theorem 2, we may deduce that each of \( A_0, \ldots, A_{p-1} \) has a unique extremal eigenvalue, and (Theorem 4) the extremal eigenvalue of block \( A_i \) has multiplicity 1. On the other hand, each extremal eigenvalue \( \lambda \) of \( A \) contributes an extremal eigenvalue \( \lambda^p \) to \( A^p \), so by pigeon-holing, there can be at most \( p \) such eigenvalues (with multiplicity) since at most one occurs in each block \( A_i \). By part (b) and a second application of Theorem 4, we know that \( A \) has at least \( p \) extremal eigenvalues; namely, \( \omega^i \rho \) for \( 0 \leq i < p \). Therefore these must be all of the extremal eigenvalues (proving (c)), and they each must occur with multiplicity 1 (proving (d)). \( \square \)

By decomposing the support graph into strongly connected components, any nonnegative matrix is permutation-equivalent to a block-triangular matrix whose diagonal blocks are strongly connected. Therefore,

**Corollary 9.** Every extremal eigenvalue of a nonnegative real matrix is an \( m \)-th root of a real number for some \( m \).

In the proof of Theorem 8, we saw that if \( A \) has periodicity \( p \), then the diagonal blocks of all sufficiently high powers of \( A^p \) are positive.

**Corollary 10.** All sufficiently high powers of a nonnegative matrix \( A \) are positive if and only if \( A \) is strongly connected and has periodicity 1.

A further consequence of Theorem 8 is that we can detect the periodicity of a nonnegative strongly connected matrix from its spectrum.

**Corollary 11.** A nonnegative strongly connected matrix \( A \) has periodicity \( p < \infty \) if and only if it has \( p \) distinct (nonzero) extremal eigenvalues.

### 3. Monotonicity

Recall if \( A \) is not strongly connected, then \( A \) can be permuted into the block triangular form \[ \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \]. In particular, the eigenvalues of \( A \) are those of \( A_{11} \) and \( A_{22} \), so we may perturb the entries of \( A_{12} \) arbitrarily without affecting the spectral radius of \( A \).

\(^2\)OK, I lied when I said that the only non-trivial thing we would take for granted would be Jordan Canonical Form. Still, you could argue that Schur’s theorem is actually pretty well-known and easy. In any case, there is a nice elementary proof that can be found in Section 3.15 of [W].
On the other hand, the following result shows that if we increase or decrease any single entry of a nonnegative strongly connected matrix (but keep the matrix nonnegative), then the spectral radius necessarily changes, and in the same direction as the perturbation.

**Theorem 12.** If $|B| \leq A$ entry-wise, then $\rho(B) \leq \rho(A)$. Moreover, if equality occurs and $A$ is strongly connected, then $|B| = A$.

**Proof.** We have $|B^n| \leq |B|^n \leq A^n$, so the entries of $B^n$ are asymptotically dominated by those of $A^n$. In particular, if $\rho(A) = 0$, then $A$ is nilpotent, and therefore $B$ must be as well. Otherwise, we may assume $\rho = \rho(A) > 0$, in which case the entries of $B^n$ are $O(n^{r-1}\rho^n)$ for some $r \geq 1$ by Lemma 1(a). If $\lambda$ is an extremal eigenvalue of $B$ and $v$ is an associated eigenvector, then some coordinates of $B^nv = \lambda^nv$ must grow at the asymptotic rate of $\rho(B)^n$, so this is possible only if $\rho(B) \leq \rho$.

In the case of equality, we may assume $B$ is nonnegative (i.e., $B = |B|$). Given that $A$ is strongly connected, Theorem 4 implies that $A$ has a positive left eigenvector $w^T$ with eigenvalue $\rho$, and Lemma 3 implies that $B$ has a nonnegative right eigenvector $v$ with eigenvalue $\rho$. It follows that

$$w^T(A - B)v = (w^TA)v - w^T(Av) = \rho w^Tv - \rho w^Tv = 0.$$  

However, $A \geq B$ and $w^T$ is positive. So if $v$ is positive, this forces $A = B$ and we are done.

Permuting coordinates if necessary, the remaining possibility is that the first $a$ coordinates of $v$ are positive, and the remaining $b$ coordinates are 0. In that case, as noted previously in the proof of Theorem 4, the condition $Bv = \rho v$ forces the southwest $b \times a$ submatrix of $B$ to vanish. Furthermore, the corresponding submatrix of $A$ cannot be zero, otherwise $A$ would not be strongly connected. Therefore, one or more of the first $a$ columns of $w^T(A - B)$ is positive, and hence $w^T(A - B)v > 0$, a contradiction. □

**References**

URL: ⟨http://www.math.upenn.edu/~wilf/DownldGF.html⟩.