M454 - Boundary Value Problems - Winter 2008

Assignment # 1.
Due: Thursday, January 17, 2008.

1. Solve each of the following ODEs for $y(x)$:

   (a) $y'' + 16y = 0$ with $y(0) = 1$ and $y'(0) = 0$.

   (b) $y'' + 6y' + 9y = 0$ with $y(0) = 1$ and $y'(0) = 0$.

   (c) $y''' - y'' + y' - y = 0$ with $y(0) = 1$ and $y'(0) = y''(0) = 0$.

2. Integration by parts allows us to transfer the derivative from one part of the integrand to another:
$$\int_a^b f(x)\,g'(x)\,dx = \Big[f(x)\,g(x)\Big]_a^b - \int_a^b f'(x)\,g(x)\,dx.$$
   Use the product rule to prove that the above statement is true.

3. A linear PDE can be written in differential operator notation $\mathcal{L}(u) = f$, where $\mathcal{L}$ is the linear differential operator, u is the unknown function, and f is the right-hand side function. For each of the following PDEs, determine the linear operator and the right-hand side function, the order of the PDE, and whether the PDE is homogeneous or nonhomogeneous:

   (a) $u_{xxx} + u_{yyy} - u = 0$

   (b) $u_{tt} - u_{xx} + u_{yy} + u_{zz} = xyz$

   (c) $x^2 u_{xx} - y^2 u_y = \cos(x) - \sin(y)$

   (d) $y^2 u_{xx} - x^2 u_y = \cos(y) - \sin(x)$

   (e) $u_t - \cos(xt)u_{xxx} - t^5 = t^2 u$

4. The following convection-diffusion-decay equation appears in many physical applications:
$$u_t = Du_{xx} - cu_x - \lambda u\,.$$
   Show that this equation can be transformed into a heat equation for $w(x,t)$ by applying the transformation
$$u(x,t) = w(x,t)e^{\alpha x - \beta t}\,.$$
   Hint: You will only obtain a heat equation for $w(x,t)$ with an appropriate choice for the constants α and β in terms of the constants D, c, and λ. Determine the choice for α and β that produces a heat equation for $w(x,t)$.

5. Consider the heat equation:
$$\begin{aligned} u_t &= \big(K_0(x)\,u_x\big)_x \\ u(0,t) &= 0 \\ u(1,t) &= 1\,, \end{aligned}$$
   where $K_0(x) = K_0(x) = e^x/\cos(x)$.

(a) Determine the steady-state solution.

(b) Plot the steady-state solution in MATLAB. Always clearly label all plots.

6. Consider the heat equation:

$$\begin{aligned} u_t &= \big(K_0(x)\, u_x\big)_x + Q(t) \\ u(1,t) &= 0 \\ u(2,t) &= 1\,, \end{aligned}$$

where $K_0(x) = x^2$.

(a) Under what condition on the heat source $Q(t)$ does a steady-state solution exist for this problem? Clearly explain your answer.

(b) Under this condition, determine the steady-state solution.

7. Consider the function:

$$u(x,t) = \sin(4\pi x)\, e^{-\pi t}\,.$$

(a) Plot this function in MATLAB over the domain $(x,t) \in [0,1] \times [0,1]$ using the `mesh` command. Always clearly label all plots. (Consult page 4 of the "Introduction to Plotting with MATLAB" guide.)

(b) Explain what you observe.