M454 - Boundary Value Problems - Winter 2008

Assignment # 3.
Due: Thursday, January 31, 2008.

1. Compute the following integrals where m and n are non-negative integers. Look out for special cases.

   (a) $\int_0^L \cos(n\pi x/L)\cos(m\pi x/L)\,dx$

   (b) $\int_0^L \cos(n\pi x/L)\sin(m\pi x/L)\,dx$

   (c) $\int_{-L}^L \cos(n\pi x/L)\sin(m\pi x/L)\,dx$

2. Consider the boundary value problem:

$$\begin{aligned} \text{PDE:} \quad & u_t = u_{xx}, \ (0 < x < 4) \\ \text{BCs:} \quad & u_x(0,t) = -2, \ u_x(4,t) = -2 \end{aligned}$$

$$\text{ICs:} \quad u(x,0) = \begin{cases} 0 & \text{if } 0 \le x \le 2 \\ x - 2 & \text{if } 2 \le x \le 4. \end{cases}$$

   (a) Do you expect a steady state solution to exist? Explain your answer.

   (b) Find the steady-state solution. (Note: You will find that the steady-state solution contains an arbitray constant. To determine the constant, show that when the net heat flux into the domain is 0, the total heat energy does not depend on time. Choose the constant in the steady-state solution so that the total energy as $t \to \infty$ is the same as the total energy at time $t = 0$.)

   (c) Using separation of variables, find the solution $u(x,t)$ to this problem. In order to accomplish this do the following:

      i. Write $u(x,t) = w(x) + v(x,t)$, where $w(x)$ is the particular solution (also the *steady-state* solution) and $v(x,t)$ is the homogeneous solution.

      ii. Write down the PDE and the BCs that $v(x,t)$ must satisfy.

      iii. Write down the solution for $v(x,t)$ using the separation of variables results we obtained in class.

      iv. Choose the arbitrary constants in $v(x,t)$ so that $u(x,t)$ satisfies the initial condition.

   (d) Plot $u(x,t)$ as a function of x for several values of t in order to see how the temperature profile evolves from the inital condition towards the steady-state solution. (**NOTE:** use the `subplot` command in MATLAB in order to save paper. In each `subplot` window plot $u(x,t)$ at a given instant in time.)

3. Use separation of variables to find the solution, in the form of an infinite series, of the homogeneous heat conduction problem with mixed boundary conditions:

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad (0 < x < L, t > 0) \\ \text{BCs:} \quad & u(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, \quad (t > 0) \\ \text{ICs:} \quad & u(x,0) = f(x) \quad (t = 0) \end{aligned}$$

Proceed as follows:

(a) Assume $u(x,t) = \phi(x)G(t)$ and derive the ODEs satisfied by $\phi(x)$ and $G(t)$.

(b) Solve the ODEs for $\phi(x)$ and $G(t)$, and determine the allowed values for the separation constant λ.

(c) Show that the eigenfunctions of the spatial eigenvalue-eigenfunction problem are mutually orthogonal.

(d) Write the solution in terms of an infinite series with coefficients B_n, and derive a formula for the B_n in terms of an integral involving the intial condition $u(x,0) = f(x)$.

4. Find the solution of

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad (0 < x < L, t > 0) \\ \text{BCs:} \quad & \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \quad (t > 0) \end{aligned}$$

for each of the following initial conditions:

(a) $u(x,0) = 4 + 3\cos\frac{4\pi x}{L}$

(b) $u(x,0) = \begin{cases} 0 & \text{if } 0 \le x < L/2 \\ 1 & \text{if } L/2 \le x \le L \end{cases}$