M454 - Boundary Value Problems - Winter 2008

Assignment # 5.
Due: Thursday, February 14, 2008.

Fourier Series

1. The integration-by-parts formula

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\,dx$$

is valid for functions $u(x)$ and $v(x)$ are continous and have a continuous first derivative.

Assume that $u(x)$, $v(x)$, $u'(x)$, and $v'(x)$ are all continuous for $a \le x < c$ and $c < x \le b$, but that all may have a jump discontinuity at $x = c$.

   (a) Derive an expression for $\int_a^b u(x)v'(x)\,dx$ in terms of $\int_a^b u'(x)v(x)\,dx$.

   (b) Show that the expression in (a) reduces to the standard integration-by-parts formula if $u(x)$ and $v(x)$ are continuous across $x = c$ (even if $u'(x)$ and $v'(x)$ are discontinuous across $x = c$).

2. Suppose that $f(x)$ is continuous except for a jump discontinuity at $x = c$: $f(c^-) = \alpha$ and $f(c^+) = \beta$ and that $f(0) = f(L) = 0$. Also assume that $f'(x)$ is piecewise smooth. Determine the Fourier cosine series of $f'(x)$ in terms of the Fourier sine series coefficients of $f(x)$.

3. The Fourier series of the function $f(x) = \cos(ax)$ on the interval $[-\pi, \pi]$, when a is not an integer, is given by

$$\cos(ax) = \frac{2a\sin(a\pi)}{\pi}\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2 - a^2}\cos(nx)\right] \quad \text{for} \quad -\pi \le x \le \pi.$$

   (a) Differentiate both sides of this equation with respect to x, differentiating the series term by term, to find the Fourier series for $\sin(ax)$:

$$\sin(ax) = -\frac{2\sin(a\pi)}{\pi}\left[\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2 - a^2}\sin(nx)\right] \quad \text{for} \quad -\pi < x < \pi.$$

   (b) Explain why this method for computing the Fourier series is valid.

   (c) If you know the Fourier series for $\sin(ax)$ given in (a), why can you not differentiate it term by term with respect to x to derive the Fourier series for $\cos(ax)$.

(d) Now consider the Fourier series of $\sin(ax)$ given in (a) as known. Explain why it can be integrated term by term to get the Fourier expansion of $\cos(ax)$.

(e) Carry out this term by term integration from 0 to x, and use it to show that

$$A_0 = \frac{\sin(a\pi)}{a\pi} = 1 + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-a^2}.$$

4. Consider the function $f(x)$ defined on the interval $[0, L]$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < a \\ \beta/\epsilon & \text{if } a < x < a+\epsilon \\ 0 & \text{if } a+\epsilon < x \leq L, \end{cases}$$

for some $a > 0$, $\epsilon > 0$ so that $a + \epsilon < L$.

(a) Find the Fourier sine series for $f(x)$.

(b) As ϵ decreases (keeping β constant), the region where $f(x)$ is nonzero shrinks, but the area under the curve remains constant. What happens to the Fourier sine series when we take the limit as $\epsilon \to 0$, with β constant.

5. Let c_n be the coefficients of the complex Fourier series of $f(x)$. Show that if $f(x)$ is a real-valued function, then $c_{-n} = \bar{c}_n$.

6. Consider the function $f(x) = 6x^2 - 1$ on $0 \leq x \leq 1$.

(a) Compute the Fourier sine series of $f(x)$. What is the smoothness of the odd extension of $f(x)$ and how quickly do the Fourier sine coefficients decay as $n \to \infty$?

(b) Compute the Fourier cosine series of $g(x) = \int_0^x f(\xi)\, d\xi$. What is the smoothness of the even extension of $g(x)$ and how quickly do the Fourier cosine coefficients decay as $n \to \infty$?

(c) Compute the Fourier sine series of $h(x) = \int_0^x g(\xi)\, d\xi$. What is the smoothness of the odd extension of $h(x)$ and how quickly do the Fourier sine coefficients decay as $n \to \infty$?