M454 - Boundary Value Problems - Winter 2008

Assignment # 7.
Due: Thursday, March 13, 2008.

The Wave Equation

1. Find the solution of the following *damped* (nonhomogeneous) wave equation:

$$\begin{aligned} &u_{tt} + bu_t = c^2 u_{xx} \quad \text{for} \quad 0 < x < L, t > 0 \\ &u(0,t) = 1, \;\; u(L,t) = 0 \\ &u(x,0) = g(x), \;\; u_t(x,0) = h(x), \end{aligned}$$

   where b, c, and L are positive constants, and g and h are unspecified functions. You may assume that b is small, $b < 2\pi c/L$.

2. Consider the following problem with Neumann boundary conditions:

$$\begin{aligned} \text{PDE:} &\quad u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \\ \text{BCs:} &\quad u_x(0,t) = u_x(L,t) = 0 \\ \text{ICs:} &\quad u(x,0) = f(x), \quad u_t(x,0) = g(x). \end{aligned}$$

   Using separation of variables show that the solution to this problem can be written in the following form:

$$u(x,t) = \frac{1}{2}\Big(F(x-ct) + F(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi.$$

   What are $F(x)$ and $G(x)$?

Sturm-Liouville

3. Consider the non-Sturm-Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\,\phi = 0.$$

   Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\,\phi = 0.$$

   Given $\alpha(x)$, $\beta(x)$, and $\gamma(x)$, what are $p(x)$, $\sigma(x)$, and $q(x)$.

4. Consider the eigenvalue problem

$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0 \quad \text{with} \quad \phi(1) = \phi(b) = 0.$$

(a) Use the result from the previous problem to put this in Sturm-Liouville form.

(b) Using the Rayleigh quotient, show that $\lambda \geq 0$.

(c) Solve this equation subject to the boundary conditions and determine the eigenvalues and eigenfunctions. Is $\lambda = 0$ an eigenvalue? Show that there is an infinite number of eigenvalues with a smallest, but no largest.

(d) The eigenfunctions are orthogonal with what weight according to Sturm-Liouville theory? Verify the orthogonality using properties of integrals.

(e) Show that the n^{th} eigenfunction has $n-1$ zeros in $1 < x < b$.

5. Consider

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + \alpha u,$$

where c, ρ, K_0, and α are functions of x, subject to

$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = f(x).$$

Assume that the appropriate eigenfunctions are known.

(a) Using the Rayleigh quotient, show that the eigenvalues are positive if $\alpha < 0$.

(b) Solve the initial value problem.

(c) Discuss the limit as $t \to \infty$.

6. Consider the fourth-order linear differential operator:

$$\mathcal{L} = \frac{d^4}{dx^4}.$$

(a) Show that $u\mathcal{L}(v) - v\mathcal{L}(u)$ is an exact differential.

(b) Evaluate $\int_0^1 [u\mathcal{L}(v) - v\mathcal{L}(u)]\, dx$ in terms of the boundary data for any function u and v.

(c) Show that $\int_0^1 [u\mathcal{L}(v) - v\mathcal{L}(u)]\, dx = 0$ if u and v are any two functions satisfying the boundary conditions

$$\phi(0) = \phi(1) = 0$$
$$\phi'(0) = \phi''(1) = 0.$$

(d) For the eigenvalue problem (using the boundary conditions from part (c))

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0,$$

show that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weighting function?

(e) Show that the eigenvalues in part (d) satisfy $\lambda \leq 0$. Is $\lambda = 0$ an eigenvalue?