M454 - Boundary Value Problems - Winter 2008

Assignment # 9.
Due: Tuesday, April 1, 2008.

1. Consider

$$\frac{d}{dx}\left((1+x)^2\frac{d\phi}{dx}\right)+\lambda(1+x)\phi=0$$

subject to $\phi(0)=0$ and $\phi(1)=0$.

(a) Using the formula derived in class (also in the book), find the asymptotic expression for the eigenvalues when $\lambda \gg 1$.

(b) Using the formula derived in class (also in the book), find the asymptotic expression for the eigenfunctions when $\lambda \gg 1$.

(c) In MATLAB plot the asymptotic expression for the eigenfunctions for $n=1$, $n=2$, $n=3$, and $n=4$.

2. Consider for $\lambda \gg 1$

$$\frac{d^2\phi}{dx^2}+\left[\lambda\sigma(x)+q(x)\right]\phi=0$$

subject to $\phi(0)=0$ and $\phi(L)=0$.

(a) Substitute

$$\phi(x)=A(x)\,\exp\left[i\sqrt{\lambda}\int_0^x\sqrt{\sigma(s)}\,ds\right]$$

into the ODE and determine a differential equation for $A(x)$.

(b) Let

$$A(x)=A_0(x)+\lambda^{-1/2}A_1(x)+\cdots.$$

Solve for $A_0(x)$ and $A_1(x)$. Verify that by only using $A_0(x)$ (i.e., neglecting the $\lambda^{-1/2}A_1(x)$ term for $\lambda \gg 1$) we get exactly the expression for $\phi(x)$ given in class. The $\lambda^{-1/2}A_1(x)$ term improves our formula.

3. Consider the Fourier sine series for

$$f(x)=\begin{cases}x & \text{if} \quad x\le 1/2\\ 1-x & \text{if} \quad x\ge 1/2\end{cases}$$

on the interval $0\le x\le 1$.

(a) How many terms in the series should be kept so that the relative mean-square error:

$$\text{RelError}(M) = \left| \int_a^b f^2 \, dx - \sum_{n=1}^{M} a_n^2 \int_a^b \phi_n^2 \, dx \right| \Big/ \left| \int_a^b f^2 \, dx \right|$$

is 10^{-4}? 10^{-5}? 10^{-6}? 10^{-7}?

(b) Based on your answers for part (a), how does the relative mean-square error depend on the number of terms in the Fourier series expansion? (Hint: assume a relationship of the form RelError $\propto M^p$ and estimate p by looking at your data.)

(c) In MATLAB plot

$$\left| f(x) - \sum_{n=1}^{M} a_n \phi_n(x) \right|$$

for the four values of M found in part (a).

4. Assuming that the operations of summation and integration can be interchanged, show that if

$$f = \sum_{n=1}^{\infty} \alpha_n \phi_n \quad \text{and} \quad g = \sum_{n=1}^{\infty} \beta_n \phi_n,$$

then for normalized eigenfunctions the following identity is true:

$$\int_a^b f g \sigma \, dx = \sum_{n=1}^{\infty} \alpha_n \beta_n,$$

a generalization of Parseval's equality.

5. Consider the following *forced* wave equation on $0 < x < \pi$, $t > 0$:

$$\begin{aligned} \textbf{PDE:} \quad & u_{tt} = u_{xx} + \cos(3x/2)\sin(3t/2) \\ \textbf{IC:} \quad & u(x,0) = u_t(x,0) = 0 \\ \textbf{BCs:} \quad & u_x(0,t) = u(\pi,t) = 0. \end{aligned}$$

(a) Solve this problem using the method of eigenfunction expansions.

(b) What happens to this solution as $t \to \infty$.