M454 - Boundary Value Problems - Winter 2008

Assignment # 10.
Due: Tuesday, April 8, 2008.

1. Consider the following reaction-diffusion equation in the rectangular domain $0 < x < 1$ and $0 < y < 1$:

$$\begin{aligned} \textbf{PDE:} \quad & u_t = u_{xx} + u_{yy} + \pi^2 u \\ \textbf{BC 1:} \quad & u_x(0, y, t) = 0 \quad \text{and} \quad u_x(1, y, t) = 0 \\ \textbf{BC 2:} \quad & u(x, 0, t) = 0 \quad \text{and} \quad u(x, 1, t) = 0. \end{aligned}$$

   (a) Solve this problem for a general initial condition:

$$\textbf{IC:} \quad u(x, y, 0) = \alpha(x, y),$$

   using separation of variables.

   (b) What happens as $t \to \infty$? Think about this carefully.

   (c) Determine all of the generalized Fourier coefficients by applying the initial condition:

$$\textbf{IC:} \quad u(x, y, 0) = \begin{cases} 1 & \text{if} \quad 1/4 < x, y < 3/4 \\ 0 & \text{otherwise.} \end{cases}$$

   (d) Plot your solution in MATLAB using the `mesh` command at

$$t = 0,\ 0.01,\ 0.02,\ 0.03,\ 0.04,\ 0.05,\ 0.06,\ 0.07.$$

   Sum up to $n = 75$ and $m = 75$. Use 50 grid points in each spatial direction.

2. Consider the following operator:

$$\mathcal{L}(\phi) = \nabla^2 \phi = \phi_{xx} + \phi_{yy},$$

on the domain Ω with boundary conditions $\phi = 0$ on $\partial\Omega$. Furthermore, let u and v be any two continuous functions that satisfy $u = v = 0$ on $\partial\Omega$.

   (a) Show that

$$u\mathcal{L}(v) - v\mathcal{L}(u) = \nabla \cdot (u\nabla v - v\nabla u).$$

(b) Show that

$$\iint_{\Omega} [u\mathcal{L}(v) - v\mathcal{L}(u)] \, dx \, dy = 0,$$

thereby proving that $\mathcal{L} = \nabla^2$ is a self-adjoint operator.

3. The vertical displacment of a non-uniform membrane satisfies

$$u_{tt} = c^2 \left(u_{xx} + u_{yy} \right),$$

where $c = c(x, y)$. Suppose that $u = 0$ on the boundary of an irregularly shaped membrane.

(a) Let

$$u(x, y, t) = \phi(x, y)h(t).$$

Show that $\phi(x, y)$ satisfies the eigenvalue problem

$$\nabla^2\phi + \lambda\sigma(x, y)\phi = 0 \quad \text{with} \quad \phi = 0 \quad \text{on the boundary.}$$

What is $\sigma(x, y)$?

(b) Prove that eigenfunctions belonging to different eigenvalues are orthogonal. (Hint: use the fact that $\mathcal{L} = \nabla^2$ is self-adjoint.)

(c) Prove that all the eigenvalues are real. (Hint: use the fact that $\mathcal{L} = \nabla^2$ is self-adjoint.)

(d) Prove that $\lambda > 0$. (Hint: derive a Rayleigh Quotient and use it.)