Math 471 - Introduction to Numerical Methods - Summer 2018

Assignment # 5.
Due: Saturday, July 21, 2018.

1. Root Finding:
   Consider \( f(x) = x^3 - 2 \).

   (a) Show that \( f(x) \) has a root \( \alpha \) in the interval \([1, 2]\).
   (b) Compute an approximation to the root by taking 4 steps of the bisection method.
   (c) Repeat, using Newton’s method. Take \( x_0 = 1.5 \) for the starting value.

   For each method, present your results in the form of a table. For the bisection method, tabulate the interval \([a_n, b_n]\), the midpoint \( x_{n+1} \), \( f(x_{n+1}) \) and the error \( |x_{n+1} - \alpha| \). For Newton’s method, tabulate \( x_n, f(x_n) \) and the error \( |x_n - \alpha| \). Discuss your results.

2. Fixed-Point Iteration:
   Which of the following iterations \( x_{n+1} = g(x_n) \) will converge to the indicated fixed point \( \alpha \) (provided \( x_0 \) is sufficiently close to \( \alpha \))? If it does converge, give the order of convergence; for linear convergence, compute \( g'(\alpha) \). In the case that \( g'(\alpha) = 0 \), expand \( g(x) \) in a Taylor polynomial about \( x = \alpha \) to determine the order of convergence.

   (a) \( x_{n+1} = -16 + 6x_n + \frac{12}{x_n}, \quad \alpha = 2 \)
   (b) \( x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}, \quad \alpha = 3^{1/3} \)
   (c) \( x_{n+1} = \frac{12}{1 + x_n}, \quad \alpha = 3 \)

   Ill-behaved root-finding:

3. Consider the function \( f(x) = \tan(x) - x \).

   (a) Use Newton’s method to find the root near \( x = 101 \). You will observe that this root is difficult to find. Starting with \( x_0 = 101 \) is not a good initial guess. Use a graphical method to determine roughly where \( \alpha \) is, then choose an initial condition \( x_0 \) sufficiently close to \( \alpha \) in order to achieve convergence. To explain the difficulty compute the quantity \( M \approx \frac{1}{2} \frac{|f''(\alpha)|}{|f'(\alpha)|} \), and refer to the discussion in class concerning \( M \). Discuss your findings.

   (b) Reformulate the problem of finding a root of \( f(x) \) by finding a function \( h(x) \) whose roots are identical to those of \( f(x) \) (hint: use the fact that \( \tan x = \frac{\sin x}{\cos x} \)). Apply Newton’s method to the \( h(x) \) that you found with \( x_0 = 101 \). Comment on the convergence in this case as compared to the findings in part a).

4. Solve the equation \( x^3 - 3x^2 + 3x - 1 = 0 \) using Newton’s method with initial guess \( x_0 = 1.001 \). Discuss the convergence of Newton’s method for this problem.
5. In our analysis of Newton’s method we showed that if \( f'(\alpha) \neq 0 \) (i.e. \( \alpha \) is a simple root), then second order convergence results. However, if \( \alpha \) is a multiple root of \( f(x) \) of multiplicity \( p \) then
\[
f(\alpha) = f'(\alpha) = f''(\alpha) = ... = f^{(p-1)}(\alpha) = 0.
\]
In this case, we can write
\[
f(x) = (x - \alpha)^p h(x)
\]
for some function \( h(x) \), and \( h(\alpha) \neq 0 \).

(a) Write the iteration function for Newton’s method in this case and evaluate \( g'(\alpha) \) (note: it will involve \( h(x) \) and \( h'(x) \)).

(b) What is the rate of convergence of Newton’s method in this case?

(c) Discuss again the convergence of Newton’s method in Problem 4.

6. Root of Nonlinear Systems:
Write down Newton’s method to solve the system \( x^2 + y^2 = 4 \), \( x^2 - y^2 = 1 \). Perform one step of Newton’s method with initial guess \( x_0 = 1, y_0 = 1 \).