

3. Consider the 2-point BVP

$$u'' = e^x \quad u'(0) = u(1) = 0$$

I. Find the exact solution

II. Set up the FD scheme described in problem (2) and use Gaussian Elimination to solve the resulting linear system. Use 20 grid points ($h = 0.05$). Compute the FD solution with discrete boundary condition (i) and (ii). Compare with the exact solution. Plot exact and computed solution on the same graph.

III. Mesh Refinement (convergence): Compute $\|E\|_\infty$ for a sequence of increasingly refined grids $h = 0.05, 0.025, 0.0125, 0.00625$. Plot $\log(\|E\|_\infty)$ against $\log(h)$. Do this for both bc (i) and (ii). Discuss the accuracy of your results.

4. Consider the 2-point BVP $u'' = f$ this time with Neumann bc's $u'(0) = \sigma_1$, $u'(1) = \sigma_2$. In class, we discussed the nonuniqueness of solutions to this problem, and argued why solutions may even fail to exist. We saw that the FD approximation results in a singular matrix.

I. What condition do $f(x)$, σ_1 and σ_2 need to satisfy for solutions to exist?

II. How does this condition relate to the requirement that the right hand side vector of the corresponding linear system must lie in the range of the matrix in order for discrete approximations to have solutions?

5. In class we considered a symmetric difference scheme for the BVP with variable coefficients $(k(x)u')' = f$, $u(0) = \alpha$, $u(1) = \beta$.

$$\frac{1}{h^2} \left(k_{j-1/2} U_{j-1} - (k_{j-1/2} + k_{j+1/2}) U_j + k_{j+1/2} U_{j+1} \right) = f_j$$

I. Show that the coefficient matrix for this approximation is negative definite provided $k(x) > 0$.

II. Show that when $f = 0$ solutions of the difference scheme satisfy a maximum principle: the value of the numerical solution U_j lies between α and β so the maximum/minimum of U_j occur on the boundary. (hint: show U_j is a convex combination of its neighbours and conclude that solution extrema can only occur on the boundary).