

M572 - Numerical Methods for Scientific Computing II - 2008

Assignment # 3.

Due: February 5, 2008.

1. L_∞ stability

In class, we discussed the stability of the centered-difference approximation for the BVP $u'' = f$ with Dirichlet BCs, and showed that $E = A^{-1}\tau$, implying $\|E\| \leq \|A^{-1}\|\|\tau\|$. To prove $O(h^2)$ convergence in L_∞ , we need to show that $\|A^{-1}\|_\infty \leq C$. In this problem, you are asked to fill in the details of the stability proof.

I. Show that $A^{-1}e_j$ extracts the j^{th} column of the matrix A^{-1} .

II. Show that the solution for $Av = e_j$ ($v = A^{-1}e_j$ is the j^{th} column of A^{-1}) is the vector obtained by evaluating the Green's function $hG(x; x_j)$ on the grid points $x_i = ih$, where

$$G(x; \bar{x}) = \begin{cases} (\bar{x} - 1)x & x \leq \bar{x} \\ (x - 1)\bar{x} & x \geq \bar{x} \end{cases}$$

III. Consider the matrix G whose elements are $G_{ij} = hG(x_i; x_j)$. Show that each element of G is bounded by h .

IV. Show that $\|A^{-1}\|_\infty = \|G\|_\infty \leq 1$

2. Nonlinear Pendulum.

Solve numerically the nonlinear pendulum problem $\theta'' = -\sin \theta$, with bc's $\theta(0) = \theta(2\pi) = 0.7$. Set up the discrete nonlinear system and solve by Newton's method. Start with initial guesses

(i) $\theta^{(0)} = 0.7$,

(ii) $\theta^{(0)} = 0.7 \cos t + 0.5 \sin t$.

Confirm the method converges quadratically.

(iii) On physical grounds, this problem does not have a unique solution. Try to find another solution. Find an initial guess $\theta^{(0)}$ that converges to this solution.

3. Elliptic equations.

I. Find the coefficient matrix A for the 5-point Laplacian operator in red-black ordering.

II. Show that the e-vectors and e-values of the 5-point Laplacian are

$$\begin{aligned}(r^{p,q})_{i,j} &= \sin(p\pi ih) \sin(q\pi jh), \\ \lambda^{p,q} &= \frac{2}{h^2} ((\cos(p\pi h) - 1) + \cos(q\pi h) - 1).\end{aligned}$$

4. Singular perturbations and boundary layers.

Consider the BVP $-\epsilon U'' + U = 1$, $U(0) = U(1) = 0$.

I. Find the exact solution.

II. Compute the solution numerically for $\epsilon = 0.001$. Use $h = 0.01$. Compare with exact solution. Discuss the accuracy of your results. Where does the maximum error occur?

III. What type of grids might be more appropriate for problems involving boundary or internal layers? Why? Design a discretization scheme on irregular grids for this problem. Design a numerical grid better suited for the above problem.

IV. BONUS: Compute the numerical solution on this grid, compare with exact solution and discuss accuracy.