

M572 - Numerical Methods for Scientific Computing II - 2009

Assignment # 4.

Due: February 19, 2009.

1. Consider the linear system

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

I. Set up the iteration matrices, G_J and G_{GS} . Compute $\rho(G)$ and $\|G\|_2$.

II. Perform 3 iterations by hand for each method. Compute the error in each solution component. Compute the ratio $\|e^{(n+1)}\|_\infty / \|e^{(n)}\|_\infty$. (Set up a table with n , $x_1^{(n)}$, $x_2^{(n)}$, $e_1^{(n)}$, $e_2^{(n)}$, $\|e^{(n)}\|_\infty$ and $\|e^{(n+1)}\|_\infty / \|e^{(n)}\|_\infty$). Discuss the rate of convergence.

III. Set up the iteration matrix for SOR. Show that if $\rho(G) < 1$, then $0 < \omega < 2$.

IV. Compute $\rho(G)$ as a function of ω (You may use Matlab to compute $\max |\lambda_k(\omega)|$). Plot $\rho(G)$ vs. ω and find ω_{opt} . Compare with the theoretical prediction.

V. Perform 3 SOR iterations by hand using ω_{opt} . Compute the error. Compute the ratio $\|e^{(n+1)}\|_\infty / \|e^{(n)}\|_\infty$. (Set up a table with n , $x_1^{(n)}$, $x_2^{(n)}$, $e_1^{(n)}$, $e_2^{(n)}$, $\|e^{(n)}\|_\infty$ and $\|e^{(n+1)}\|_\infty / \|e^{(n)}\|_\infty$). Discuss the rate of convergence.

2. Show that if A is strictly diagonally dominant, $\|G_J\| < 1$ (in which norm?), and therefore the iteration converges.

3. Aitken's acceleration. Assume that the iteration matrix G has a complete set of e-vectors, r_k , with corresponding e-values λ_k . Assume that λ_1 is the largest in magnitude. The set of e-vectors can be used as a basis.

I. By expanding the initial error $e^{(0)}$ in terms of the basis r_k , show that $e^{(n+1)} \approx \lambda_1 e^{(n)}$.

II. From the above, one has

$$\begin{aligned} e^{(n+1)} &\approx \lambda_1 e^{(n)} \\ e^{(n)} &\approx \lambda_1 e^{(n-1)}. \end{aligned}$$

Eliminate λ_1 between the above two (approximate) equations. Replace $e^{(n)} = x - x^{(n)}$ etc. and rearrange the resulting equations to show that the exact solution $x = (x_1, \dots, x_n)^T$ satisfies

$$x_i \approx \frac{x_i^{(n+1)} x_i^{(n-1)} - (x_i^{(n)})^2}{x_i^{(n+1)} - 2x_i^{(n)} + x_i^{(n-1)}} \approx x_i^{(n+1)} - \frac{(x_i^{(n+1)} - x_i^{(n)})^2}{x_i^{(n+1)} - 2x_i^{(n)} + x_i^{(n-1)}}$$

III. Perform 3 Gauss-Seidel iterations on

$$\begin{pmatrix} 5 & 3 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}.$$

Use Aitken's acceleration to improve the Gauss-Seidel approximations.

IV. Write a small routine for Gauss-Seidel's method and check how many GS iterations would be required to achieve the same accuracy.