M572 - Numerical Methods for Scientific Computing II - 2009

Assignment # 5.
Due: March 5, 2009.

1. Consider Euler's method applied to the linear ode $u' = au + b$, $u(0) = u_0$. Find a closed form for the numerical solution u^n and show directly that the numerical solution converges to the exact solution as $k \to 0$.

2. In class, we have derived conditions for a general linear multistep method to be p^{th} order accutrate

$$\sum_{j=0}^{r} \frac{j^q}{q!}\alpha_j = \sum_{j=0}^{r} \frac{j^{q-1}}{(q-1)!}\beta_j, \qquad q = 0, 1, ..., p$$

What is the maximum order of accuracy of a general r-step LMM?

3. Find the coefficients of the 4-step Adams-Bashforth method. Give the expression for the local truncation error.

4. Check that the 4-stage Runge-Kutta method is 4^{th} order accurate for the linear case $u' = \lambda u$, by comparing the one-step behaviour of the exact and numerical solutions and showing that the one-step error is $O(k^5)$. Is this the only one?

5. Asymptotic error expansion. Consider Euler's method for $u'(t) = f(u)$, $u(0) = u_0$. The method is first order accurate, and the error can written in a power series in k. In the following, you are asked to verify the form of the leading order error term. Show that the error

$$u_n = u(t_n) + kE_n + O(k^2)$$

where $E_n = E(t_n)$ is the solution of the ODE

$$E' = f'(u)E - \frac{1}{2}u'', \quad E(0) = 0.$$

$E(t)$ is called the Principal Error Function.
Hint: define $d_n = u^n - (u(t_n) + kE_n)$. Obtain a relationship between d_{n+1} and d_n. Taylor expand about t_n to show that if E satisfiles above ODE then

$$d_{n+1} = d_n + kf'(u(t_n))d_n + O(k^3)$$

Procede as in Euler's convergence proof, to conclude that $d_n = O(k^2)$.

6. Consider the problem $u' = -u^2$, $u(0) = 1$.

I. Find the exact solution.
II. Compute $u(t)$ for $0 \leq t \leq 1$ using Euler's method with $k = 0.1,\ 0.05,\ 0.025$ and 0.0125. For each step size, plot computed and exact solutions.
III. For t=1, make a table of $k,\ u^n,\ e^n$ and e^n/k.
IV. Find the principal error function $E(t)$. Compare $E(1)$ with the last column of part III.