M572 - Numerical Methods for Scientific Computing II - 2009

Assignment # 6.
Due: March 12, 2009.

1. The trapezoid method is $u^{n+1} = u^n + \frac{k}{2}(f(u^n) + f(u^{n+1}))$.
I. Show that the LTE is $O(k^2)$.
II. Show that the method is A-stable.
III. Show directly that the method converges (i.e. that $\lim_{k\to 0} e^n = 0$. Hint: Obtain a relationship between e^{n+1} and e^n and proceed as in the convergence proof for Euler's method).

2. Solve the following difference equations.
I. $u^{n+2} - 2u^{n+1} - 3u^n = 0,\ \ u^0 = 0,\ u^1 = 1.$
II. $u^{n+3} - 3u^{n+2} + 3u^{n+1} - u^n = 0,\ \ u^0 = 1,\ u^1 = 0,\ u^2 = -3.$

3. Consider Milne's method

$$u^{n+1} = u^{n-1} + \frac{k}{3}(f(u^{n+1}) + 4f(u^n) + f(u^{n-1}))$$

I. Show that the LTE is $O(k^4)$.
II. Show that the root condition is satisfied.
III. Show that when the method is applied to $u' = \lambda u$, the roots of the characteristic polynomial satisfy

$$\zeta_1 = e^{\lambda k} + O(k^5), \quad \zeta_2 = -e^{-\frac{\lambda k}{3}} + O(k^3).$$

IV. Compute the solution of $u' = u$, $u(0) = 1$, using $k = 0.1$. Take $u^0 = u(0)$ and use the forward Euler method to compute the starting value u^1. Compute the error at $t = 5$. Repeat with $k = 0.01$. Now apply the method to $u' = -u$, $u(0) = 1$. Explain the results.

4. Derive the 3-step Backward Differentiation Method.

5. Consider the initial value problem

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} -11 & 9 \\ 9 & -11 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad , \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_0 = \begin{pmatrix} 1.0 \\ 1.2 \end{pmatrix}$$

Find the exact solution. Compute the solution for $0 \leq t \leq 2$ using the forward Euler and the backward Euler methods with step size $k = 0.15$, 0.12, 0.11, 0.10, 0.09 and 0.05. Plot the first component of the solution vs. time (use the MATLAB command subplot(221)). Explain the results.

6. Use the boundary locus method to find the absolute stability region of (i) the 2-stage RK and (ii) the 2-step BDF method.