

M572 - Numerical Methods for Scientific Computing II - 2008

Assignment # 7.

Due: March 27, 2008.

1. Consider the Crank-Nicolson scheme for the heat equation.

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{1}{2}(D_+D_-u_j^n + D_+D_-u_j^{n+1}).$$

I. Compute the LTE.

II. Show that if $r = k/h^2 \leq 1$ the scheme satisfies a maximum principle $|u_j^{n+1}| \leq \max|u_j^n|$.

III. Show that if $r \leq 1$ the method converges.

2. Consider the following difference scheme for the heat equation

$$\frac{u_j^{n+1} - u_j^n}{k} = D_+D_-u_j^n - \frac{h^2}{12}(D_+D_-)^2u_j^n.$$

I. Show that the LTE is $O(k + h^4)$.

II. Find the amplification factor $g(\xi h)$.

III. For what values of $r = k/h^2$ is the scheme stable in the l_2 norm?

3. Consider the following scheme for the heat equation

$$\frac{u_j^{n+1} - u_j^n}{k} = D_+D_-u_j^{n+1}.$$

I. Use the energy method to prove the scheme is unconditionally stable in the l_2 norm.

II. Find the amplification factor $g(\xi h)$ and show that $0 \leq g(\xi h) \leq 1$ for all $|\xi h| \leq \pi$.

4. Compute the solution of $u_t = \epsilon u_{xx}$, $\epsilon = 0.05$, $u(x, 0) = \sin \pi x$ on $[0,1]$ with boundary conditions $u(0, t) = u(1, t) = 0$. Use the second order difference approximation in space and the forward Euler and backward Euler schemes in time. Take $h = 0.05$, and $k = 0.1, 0.05, 0.025, 0.01$. Plot the computed and exact solutions at $t = 0, 0.25, 0.5, 1.0, 2.0$. (superimpose exact and numerical solutions and use subplot to fit several pictures on one page). Discuss the results.