

M572 - Numerical Methods for Scientific Computing II - 2009

Assignment # 7.

Due: April 2, 2009.

1. Given u_j , $j = 0, \pm 1, \pm 2, \dots$ define $\hat{u}(\xi h) = \frac{h}{2\pi} \sum_{j=-\infty}^{\infty} u_j e^{-ij\xi h}$. Verify the following formulas.

(a) $u_j = \int_{-\pi/h}^{\pi/h} \hat{u}(\xi h) e^{ij\xi h} d\xi$,

(b) $\int_{-\pi/h}^{\pi/h} |\hat{u}(\xi h)|^2 d\xi = \frac{h}{2\pi} \sum_{j=-\infty}^{\infty} |u_j|^2$.

2. Consider the following difference scheme for the heat equation

$$\frac{u_j^{n+1} - u_j^n}{k} = D_+ D_- u_j^n - \frac{h^2}{12} (D_+ D_-)^2 u_j^n$$

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I. Show that the LTE is $O(k + h^4)$.

II. Find the amplification factor $g(\xi h)$.

III. For what values of $r = k/h^2$ is the scheme stable in the l_2 norm?

3. Consider the following scheme for the heat equation

$$\frac{u_j^{n+1} - u_j^n}{k} = D_+ D_- u_j^{n+1},$$

corresponding to the backward Euler in time and 2nd order centered differencing in space.

I. Use the energy method to prove the scheme is unconditionally stable in the l_2 norm.

II. Find the amplification factor $g(\xi h)$ and show that $0 \leq g(\xi h) \leq 1$ for all $|\xi h| \leq \pi$.

4. Compute the solution of the heat equation $u_t = u_{xx}$, on $0 \leq x \leq 1$, with zero Dirichlet boundary conditions $u(0, t) = u(1, t) = 0$ and initial conditions (a) $u(x, 0) = \sin \pi x$; (b) $u(x, 0) = 1 - 2|x - \frac{1}{2}|$. Use centered differencing in space and forward Euler in time, with $h = 0.05$, and $k = 0.0012, 0.0013$. Plot the computed solution at $t = 0, k, 25k, 50k, 100k$. Discuss the results. (If you are using Matlab, use the command subplot to fit several figures on the same page).
5. Consider the heat equation $u_t = u_{xx}$ on $0 \leq x \leq 1$, with zero Neumann boundary conditions $u_x(0, 1) = u_x(1, t) = 0$. Consider the method of lines $U'_j = D_+ D_- u_j$ for $j = 0, \dots, m + 1$, with $D_0 u_0 = D_0 u_{m+1} = 0$ to account for the boundary conditions where $D_0 = \frac{1}{2}(D_+ + D_-)$.
- (a) Write the system in the form $U' = AU$ where $U(t) = (u_0(t), \dots, u_{m+1}(t))^T$. Find the eigenvalues and eigenvectors of the matrix A (use eigenfunctions of the corresponding continuous problem).
- (b) Suppose the forward Euler method is used to solve $U' = AU$. Show that the scheme is stable for $r \leq \frac{1}{2}$.
- (c) Consider the initial condition

$$u(x, 0) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}.$$

This problem corresponds to the diffusion of gas molecules in a closed container, where $u(x, t)$ is the gas density at point x and time t . The initial condition corresponds to a vacuum state in the left half of the container and gas at uniform density in the right half. Solve the problem numerically using the scheme described above, with $h = 0.05$ and $r = \frac{1}{2}, \frac{1}{4}$. Plot the results for $t = 0, 1, 2$. Based on the numerical results, what is the long-time asymptotic state of the system?