M572 - Numerical Methods for Scientific Computing II - 2009

Assignment # 8.
Due: April 16, 2009.

1. Consider the linear advection equation $u_t + au_x = 0$, where the constant a may be either positive or negative. The Lax-Friedrichs scheme is

$$\frac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{k} + a\frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0$$

I. Show that the scheme is stable in the 2-norm and the ∞-norm if $|\nu| < 1$.

II. Show that the scheme converges with first order accuracy if $|\nu| < 1$

2. The centered difference scheme

$$\frac{u_j^{n+1} - u_j^n}{k} + a\frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0$$

is unconditionally unstable. One way to stabilize it is by 'smoothing', that is replacing the u_j^n term in the time derivative by the average of its neighbours $\frac{1}{2}(u_{j+1}^n + u_{j-1}^n)$, which gives the Lax-Friedrichs scheme. Another way was mentioned in class: combine the centered differencing spatial approximation with a suitable (not forward Euler) time integration scheme. Analyze the accuracy and stability of the resulting scheme.

3. Show that the upwind scheme for $u_t + au_x = 0$, $a > 0$ is dissipative of order 2 if $\nu < 1$.

4. For the linear advection equation ($a > 0$), the exact value of the solution u_j^{n+1} can be obtained by tracing the characteristic back in time to t_n and use the fact that the solution remains constant along this line. In general, the characteristic will not cross at grid points but rather in between grid points. This calls for interpolation. Show that linear interpolation of the data at time t_n (together with characteristic theory) gives the upwind scheme. Show that quadratic interpolation gives the Lax-Wendroff scheme. In both cases,

identify special values of the CFL number ν for which the interpolation error vanishes (i.e. the numerical solution becomes exact).

5. Consider $u_t + u_x = 0$ with initial data $u(x,0)$ given by

$$f_1(x) = \begin{cases} 1.0 & 0 \le x < 2 \\ 0.5 & x = 2 \\ 0.0 & 2 < x \le 6 \end{cases}, \qquad f_2(x) = \begin{cases} 0.0 & 0 \le x < 1 \\ 1 - |x-2| & 1 \le x \le 3 \\ 0.0 & 3 < x \le 6 \end{cases}.$$

Compute the solution for $0 \le x \le 6$ and $t = 2$. Use the upwind, Lax-Friedrichs and Lax-Wendroff schemes with $h = 0.05$ and $k = 0.04,\ 0.06$. For each scheme, plot the numerical solution and the exact solution at time $t = 2$. Discuss the numerical results. Derive the modified equation for the Lax-Wendroff scheme and use it to explain the numerical results.

*** Bonus Problem ***

6. Consider the nonlinear advection equations

$$u_t + uu_x = 0$$

which can also be written as

$$u_t + (\frac{1}{2}u^2)_x = 0.$$

This equation is known as Burgers equation. Consider the initial data

$$u_0(x) = \begin{cases} 3 & x < 0 \\ 1 & x > 0 \end{cases}$$

Solve both formulations of this equation using a nonlinear version of the first order upwind scheme. Solve the problem on the domain $[-1,\ 1]$. Use h=0.02 (= 100 mesh points). Explain your choice of a stable time step. Keep the left hand boundary at $u = 3$. Plot both solutions at $t = 0.4$ on the same graph and discuss your results.