In this problem, we are concerned with the solution for the Riemann problem for the Euler equations. We use $\rho, u$ and $p$ to denote density, velocity and pressure, $\tau = \rho^{-1}$ is the specific volume, $c = \sqrt{\frac{\gamma p}{\rho}}$ is the speed of sound, $\gamma$ the specific heat ratio and $\mu^2 = \frac{\gamma - 1}{\gamma + 1}$.

For the Euler equations, the following relations hold across right/left moving shocks

$$u = u_0 + (p - p_0)\sqrt{\frac{(1 - \mu^2)\tau_0}{p + \mu^2 p_0}} : S_{\rightarrow}$$

$$u = u_0 - (p - p_0)\sqrt{\frac{(1 - \mu^2)\tau_0}{p + \mu^2 p_0}} : S_{\leftarrow}$$

The following relations hold across rarefactions

$$u - \frac{2}{\gamma - 1}c = \text{const} = u_0 - \frac{2}{\gamma - 1}c_0 : R_{\rightarrow}$$

$$u + \frac{2}{\gamma - 1}c = \text{const} = u_0 + \frac{2}{\gamma - 1}c_0 : R_{\leftarrow}$$

$$\frac{p}{\rho^\gamma} = \text{const} = \frac{p_0}{\rho_0^\gamma} \quad \text{(entropy = const)}$$

Across contact discontinuity

$$u = u_0 , \quad p = p_0$$

(*) Note that $\frac{p}{\rho^\gamma} = \text{const}$ can be used to obtain a relation between $u$ and $p$ across a rarefaction fan, by substituting for $c$ in the equations above.
(i) Write a program that ‘cooks up’ exact solutiond for the Riemann problem. Assume your solution structure consists of a right shock, a contact discontinuity and a left rarefaction (see figure).

(a) Begin with a given right state \( W_R = (\rho_R, u_R, p_R) = (1, 0, 1) \).
(b) Select a post shock pressure \( p^* (p^* > p_R) \) and determine the post-shock state \( W^*_R \) and the shock speed.
(c) Select \( \rho^*_L \) and determine the state \( W^*_L \) to the right of the contact discontinuity, and the speed of the contact.
(d) Select \( p_L, (p_L > p^*) \) and determine the left state \( W_L \).
(e) Determine the solution inside the rarefaction \( W = W(\xi) \), \( \xi = \frac{x}{t} = u - c \).
(f) You have now constructed a similarity solution \( W = W(\xi) \). Choose a time \( t \) and plot your solution \( W(x,t) \) corresponding to that time.

(ii) Given \( W_L \) and \( W_R \), use the shock+rarefaction relations to find \( p^* \) and \( u^* \) in the 'star'-region. This requires an iteration of a nonlinear equation in \( p^* \) (or \( u^* \), depending on how you set it up). Then compute \( \rho^*_{L,R} \) from the corresponding shock/rarefaction relations. Compute the shock speed and the solution inside the rarefaction. Run your program on the standard shock tube problem (Sod’s shock tube problem)

\[
W_L = (\rho_L, u_L, p_L) = (1, 0, 1) \quad W_R = (\rho_R, u_R, p_R) = (0.125, 0, 0.1)
\]

II. Consider the advection equation \( u_t + au_x = 0 \), and assume \( a > 0 \).

A general (explicit) 2-level numerical scheme for solving the advection equation has the following form

\[
u_{j}^{n+1} = \sum_{k=-l}^{r} c_k u_{j+k}^{n}
\]

where the right hand side (RHS) is some approximation to \( u^* \) (see figure) which in is exactly what the value of \( u_j^{n+1} \) should be. One approach to compute \( u^* \) is to use interpolation of the availables grid values \( u_j^k \). When the characteristics cross exactly at a grid point, no interpolation is required and the scheme then becomes exact. (when this happens, \( \nu = ak/h \) is an integer, here \( k \) is the time step and \( h \) the grid spacing).
1. Construct the first order upwind scheme by taking $l = 1, r = 0$ and requiring that the scheme is exact for $\nu = 0, 1$.

2. Construct the Lax-Wendroff (LW) scheme by taking $l = 1, r = 1$ and requiring the scheme is exact for $\nu = -1, 0, 1$.

3. Construct the Beam-Warming (BW) scheme by taking $l = 2, r = 0$ and requiring that the scheme is exact for $\nu = 0, 1, 2$.

4. Show that LW and BW are equivalent through the transformation $\nu = \nu + 1, j = j - 1$. 