3 Special values of \( L(s, \chi) \)

3.1 The values at negative integers: generalized Bernoulli numbers

Recall that in the previous lecture, we showed the analytic continuation and functional equation for \( L(s, \chi) \) where \( \chi \) is a primitive character of conductor \( f \). The key point was to define an analytic function

\[
H(s) = \int_C F(z) z^{s-1} \frac{dz}{z},
\]

for a certain contour \( C \), where

\[
F(z) = \sum_{a=1}^f \frac{\chi(a) z e^{az}}{e^{fz} - 1}.
\]

The analytic continuation followed from the relation (for \( \Re(s) > 0 \))

\[
L(s, \chi) = -\frac{1}{2\pi i} \Gamma(1 - s) H(s).
\]

The functional equation is of the form

\[
L(s, \chi) = \frac{\tau(\chi)}{2\pi \delta} \left( \frac{2\pi}{f} \right)^s \frac{L(1 - s, \bar{\chi})}{\Gamma(s) \cos \frac{\pi(s-\delta)}{2}},
\]

where \( \delta = 0 \) or \( 1 \) according as \( \chi \) is even or odd. We will now use these relations to study the values at integer points of \( L(s, \chi) \).

Definition 3.1. The generalized Bernoulli numbers \( B_{n,\chi} \) are defined by

\[
\sum_{a=1}^f \frac{\chi(a) t e^{at}}{e^{ft} - 1} = \sum_{n=0}^\infty B_{n,\chi} \frac{t^n}{n!}
\]

Note that the numbers \( B_{n,\chi} \) live in the field \( \mathbb{Q}(\chi) \) obtained from \( \mathbb{Q} \) by adjoining the values of the character \( \chi \).

Example 3.2. One checks easily for instance that

\[
B_{0,\chi} = \sum_{a=1}^f \frac{\chi(a)}{f},
\]

which is 0 if \( \chi \neq 1 \) and equal to 1 if \( \chi = 1 \). Also

\[
B_{1,\chi} = \sum_{a=1}^f \chi(a) a
\]

if \( \chi \neq 1 \) and \( B_{1,1} = \frac{1}{2} \).

The values of \( L(s, \chi) \) at negative integers may be expressed in terms of generalized Bernoulli numbers:
Theorem 3.3. For any integer $n \geq 1$,

$$L(1-n, \chi) = -\frac{B_{n,\chi}}{n}.$$ 

Proof. Indeed, from (1), for $n \geq 1$,

$$L(1-n, \chi) = -\frac{1}{2\pi i} \Gamma(n) H(1-n) = -\frac{\Gamma(n)}{2\pi i} \int_{C_\varepsilon} f(z) z^{-n-1} dz = -\frac{B_{n,\chi}}{n}. $$

Note that by the functional equation for the $L$-function it follows that $L(1-n, \chi) = 0$ for $n$ odd if $\chi$ is an even character (except when $\chi = 1$ and $n = 1$) and for $n$ even if $\chi$ is an odd character. In particular this implies that $B_{n,\chi} = 0$ for $\chi \neq 1$, $n \equiv \delta \mod 2$, a result that can also be easily checked directly. Also from the functional equation, one easily derives formulae for $L(n, \chi)$, $n \geq 1$ when $n \equiv \delta \mod 2$, for instance that

$$\zeta(2m) = (-1)^{m+1} \frac{2^{2m-1} \pi^{2m} B_{2m}}{(2m)!}$$

for $m \geq 1$. The values $L(n, \chi)$ for $n \geq 1, n \not\equiv \delta \mod 2$ are, however, much more mysterious. (For instance, see the next section for a discussion of $L(1, \chi)$ when $\chi$ is a nontrivial even character.)

3.2 The value $L(1, \chi)$

For odd characters we can compute $L(1, \chi)$ using the functional equation. Indeed,

$$L(1, \chi) = \frac{\tau(\chi)}{2i} \frac{2\pi}{f} L(0, \overline{\chi}) = \frac{\pi i \tau(\chi)}{f} B_{1, \chi} = \frac{\pi i \tau(\chi)}{f} \sum_{a=1}^{f} \overline{\chi}(a)a.$$

For even characters the functional equation is not useful. Of course $L(s, 1)$ has a pole at $s = 1$. If $\chi \neq 1$, we have

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{\tau(\chi)} \sum_{a=1}^{f} \overline{\chi}(a) e^{2\pi i a/f} = \frac{1}{\tau(\chi)} \sum_{a=1}^{f} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{1}{n} e^{2\pi i a/f}.$$

$$= -\frac{1}{\tau(\chi)} \sum_{a=1}^{f} \overline{\chi}(a) \log(1 - \zeta_a^{\chi}) = -\frac{\tau(\chi)}{f} \sum_{a=1}^{f} \overline{\chi}(a) \log(1 - \zeta_a^{\chi})$$

$$= -\frac{\tau(\chi)}{f} \sum_{a=1}^{f} \overline{\chi}(a) \log |1 - \zeta_a^{\chi}|.$$

We now show that for $\chi$ nontrivial $L(1, \chi) \neq 0$. (By the product formula for the $L$-function, this is obvious if $\Re(s) > 1$.) For $K$ a number field, let $\zeta_K(s) = \sum_a \frac{1}{N(a)^s} = \prod_{p} (1 - \frac{1}{N(p)^s})^{-1}$ where $a$ runs over integral ideals and $p$ over prime ideals in $\mathcal{O}_K$. It is known that $\zeta_K(s)$ admits an analytic continuation to the whole complex plane with a pole of order one at $s = 1$.

Proposition 3.4. Let $m$ be a positive integer, and let $K = \mathbb{Q}(\zeta_m)$. Then

$$\zeta_K(s) = \prod_{\chi, f|\chi|m} L(s, \chi).$$

(2)
Proof. Let $X_K$ denote the set of characters $\chi$ whose conductor divides $m$. Since we have a canonical isomorphism $G := \text{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$, $X_K$ may be identified with the set of characters of $G$. To prove the theorem, it suffices to show that for all rational primes $p$,

$$\prod_{p \mid p} (1 - Np^{-s}) = \prod_X (1 - \chi(p)p^{-s}), \quad (3)$$

where $p$ runs over the primes of $K$ lying over $p$. Fix a rational prime $p$ and let $e, f, r$ denote the ramification index, inertial degree and number of primes lying over $p$ respectively. Let $K^u$ denote the maximal subfield of $K$ in which $p$ is unramified. If $m = p^r \cdot n$ with $(p, n) = 1$, then clearly $K^u = \mathbb{Q}(\zeta_n)$. Further, $\chi(p) \neq 0$ exactly when $f | m$, or what is the same thing, the character $\chi$ factors through $\text{Gal}(K^u/K) = (\mathbb{Z}/n\mathbb{Z})^\times$. Further, $\chi(p)$ must be an $f$th root of unity, since $f$ is the order of $p$ in $(\mathbb{Z}/n\mathbb{Z})^\times$. Conversely any character on $(\mathbb{Z}/n\mathbb{Z})^\times$ can be obtained by first defining it on $\langle p \rangle \subset (\mathbb{Z}/n\mathbb{Z})^\times$ and then extending it to the whole group: the number of ways of making such an extension is just the index of $\langle p \rangle$ in $(\mathbb{Z}/n\mathbb{Z})^\times$ which is $r$. Thus for any indeterminate $t$,

$$\prod_{\chi, f | m} (1 - \chi(p)t) = \prod_{\chi, f | m} (1 - \chi(p)t) = \prod_{\zeta, \zeta^f = 1} (1 - \zeta t)^r = (1 - t^f)^r,$$

from which (3) follows by setting $t = p^{-s}$. \qed

The identity (2) can also be written as

$$\zeta_K(s) = \zeta(s) \cdot \prod_{\chi \neq 1, f | m} L(s, \chi).$$

Since both $\zeta_K(s)$ and $\zeta(s)$ have a pole of order 1 at $s = 1$, and since $L(s, \chi)$ is analytic at $s = 1$ for nontrivial $\chi$, we immediately get:

**Corollary 3.5.** If $\chi$ is nontrivial, $L(1, \chi) \neq 0$.

We note the following remarkable consequence for which there is no elementary proof known.

**Corollary 3.6.** Suppose $\chi$ is an odd character. Then $\sum_{a=1}^f \chi(a)a \neq 0$.

For future use, we state the following generalization of the proposition above, whose proof we leave as an exercise.

**Proposition 3.7.** Let $L$ be any subfield of $K$, and let $X_L$ denote the set of characters $\chi$ of conductor dividing $m$ that (viewed as characters of $G$) factor through $\text{Gal}(L/\mathbb{Q})$ i.e. are trivial on $H := \text{Gal}(K/L)$. Then

$$\zeta_L(s) = \prod_{\chi \in X_L} L(s, \chi).$$

**3.3 The connection with class numbers**

The connection between the $L$-values $L(1, \chi)$ and class numbers is a consequence of the following result, whose proof can be found in any standard text on algebraic number theory.

**Theorem 3.8.** The residue at $s = 1$ of $\zeta_K(s)$ is given by the formula

$$\text{Res}_{s=1} \zeta_K(s) = \frac{2^{r_1(K)}(2\pi)^{r_2(K)}h_KR_K}{w_K\sqrt{d_K}},$$

where

- $r_1(K)$ is the number of real embeddings of $K$.
- $r_2(K)$ is the number of complex embeddings of $K$.
- $h_K$ is the class number of $K$.
- $R_K$ is the regulator of $K$.
- $w_K$ is the number of roots of unity in $K$.
where \( r_1(k) = \) the number of real embeddings of \( K \), \( r_2(k) = \) half the number of complex embeddings of \( K \), \( R_K \) is the regulator, \( h_K \) the class number, \( w_K \) the number of roots of unity in \( K \) and \( d_K \) the discriminant of \( K \). Further if we define

\[
\Lambda_K(s) = A^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_K(s)
\]

with \( A = 2^{-r_2} \pi^{-\deg(K/Q)/2} \sqrt{|d_K|} \), then \( \Lambda_K(s) = \Lambda_K(1 - s) \).

In the case of \( L \subset Q(\zeta_m) \), this yields

Corollary 3.9.

\[
\prod_{\chi \in X_L, \chi \neq 1} L(1, \chi) = \frac{2^{r_1(L)}(2\pi)^{r_2(L)} h_L R_L}{w_L \sqrt{|d_L|}}.
\]

Exercises:

1. Prove Prop. 3.7.