Homework 2, due Feb 1

(1) Let $\zeta(s)$ denote the usual Riemann zeta function. Show that in a neighborhood of 1, the function $\log \zeta(s)$ differs from
\[
\sum_p \frac{1}{p^s}
\]
by a function that is analytic at $s = 1$. (Here $p$ varies over the rational primes.)

(2) Next, let $k$ be a number field and $\zeta_k(s)$ the zeta function of $k$. Show that $\log \zeta_k(s)$ differs from
\[
\sum_{p, \deg p = 1} \frac{1}{Np^s}
\]
by a function that is analytic at $s = 1$. (Here $p$ varies over prime ideals in $\mathcal{O}_k$.)

(3) Let $K/k$ be an extension of number fields. Show that the trace map from $K$ to $k$ is compatible with the inclusions $K \hookrightarrow \mathbb{A}_K$, $k \hookrightarrow \mathbb{A}_k$ and the trace map $\mathbb{A}_K \to \mathbb{A}_k$. Also state and prove an analog for the norm map.

(4) Let $k$ be a number field, $\mathbb{A}_k$ the adeles of $k$ and $J_k$ the ideles of $k$. Show that the topology on $J_k$ is not the subspace topology that one gets from the inclusion $J_k \hookrightarrow \mathbb{A}_k$.

(5) Consider the inclusion $J_k \hookrightarrow \mathbb{A}_k \times \mathbb{A}_k$
given by $x \mapsto (x, x^{-1})$. Show that the topology on $J_k$ is the subspace topology obtained by thinking of $J_k$ as a subspace of $\mathbb{A}_k \times \mathbb{A}_k$ (with the product topology) via the map above.