Math 631: Algebraic Geometry: Homework Set 1.

Due Friday September 12, 2008

In this problem set, all varieties (algebraic sets) are defined over a fixed algebraically closed field k (unless otherwise stated).

1. a). Let X_t be the set of $n \times m$ matrices of rank at most t, with entries in k. Prove or disprove that X_t is an algebraic subset of $k^{n \times m}$.

b). Consider the set SU_2 of 2×2 special unitary matrices. Is SU_2 an algebraic subset of $\mathbb{C}^4$? Identifying $\mathbb{C}$ with $\mathbb{R}^2$, is SU_2 an algebraic subset of $\mathbb{R}^8$? Explain. (Recall: elements of SU_2 are 2 by 2 complex matrices of determinant one whose inverse is equal to its conjugate transpose).

2. Consider the map

$$\mathbb{A}^1 \to \mathbb{A}^3$$

$$t \mapsto (t, t^2, t^3).$$

a). Show that the image V of this map is an algebraic set that can be defined by two equations.

b). Prove or disprove that your two equations generate the ideal $\mathcal{I}(V)$ of all polynomials in $k[x, y, z]$ vanishing on V.

3. a). Identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ as sets in the natural way. Show that the Zariski topology on $\mathbb{A}^2$ is *not* the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$.

b). Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be Zariski closed sets of $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively. Prove that $V \times W$ is a Zariski closed set of $\mathbb{A}^{n+m}$, by finding defining equations.

c). Prove that the coordinate ring for $V \times W$ is isomorphic to $k[V] \otimes_k k[W]$.

4. Assume the characteristic of k is not two. Let V_c be the intersection of the circle $x^2 + y^2 = 1$ in $\mathbb{A}^2$ with the line $x = c$, where c is a fixed element of k.

a). Find the ideal of polynomials in $k[x, y]$ vanishing on V. Two choices of c are special: which ones? Explain how they are special both algebraically and geometrically. (Draw a picture!)

b). Consider the projection $\mathbb{A}^2 \to \mathbb{A}^1$ sending $(x, y) \mapsto x$, restricted to the circle $x^2 + y^2 = 1$. (Draw a picture!) Interpret the variety V_c geometrically in terms of this map. The special values of c are called "ramification points": what does that mean in this set-up?

This exercise hints at the importance of *schemes*.. The "right" intersection object V_c really ought not to be a variety when c is a ramification point: why not? What should it be? What should its coordinate ring be?

c). What happens if the characteristic is two?

5. Let $\phi : \mathbb{A}^n \to \mathbb{A}^n$ be an regular map, given by polynomials $f_1, \ldots, f_n$ in variables $x_1, \ldots, x_n$. Let J be the Jacobian polynomial of ϕ, the determinant of the matrix $\partial f_i / \partial x_j$.

a). Show that if ϕ is an isomorphism, then J is a non-zero constant.

b). Show that if J is a non-zero constant, then ϕ is an isomorphism (for this, assume $k = \mathbb{C}$ and $n = 2$). (HINT: Solve this problem only after all others are done.[1])

6. Describe the image of the regular map

$$\mathbb{A}^2 \to \mathbb{A}^2$$

$$(x, y) \mapsto (x, xy).$$

Is is open? closed? dense?

7. For the morphisms described in 2, 4b, 5, and 6, explicitly describe the induced map of coordinate rings.

[1]If you solve it, I'm happy to chair your 2009 PhD dissertation committee.