Math 631: Problem Set 10

Due Friday November 21, 2008

1. Let Z be the subvariety $\mathbb{V}(x, y)$ of the quadric three-fold $X = \mathbb{V}(xw - yz)$ in affine four space.

a.) Show that Z defines a prime divisor of X.

b). Show that Z is locally principal on the open subvariety $X - \{(0, 0, 0, 0)\}$, giving an explicit representative as a Cartier divisor.

c). Show that Z is not locally principal at the origin. (Hint: use the grading.)

d). Show that X is normal by using the following theorem: *Any complete intersection whose singular locus has codimension at least two is normal.* (You do not have to prove this fact). What property does X fail to have that would have implied that every divisor is locally principal? Demonstrate how X fails to have this property.

2. Prove that the Picard group of $\mathbf{P}^m \times \mathbf{P}^n$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Explicitly describe a set of generators.

3. Let $\pi : \tilde{\mathbb{A}}^2 \longrightarrow \mathbb{A}^2$ be the blowup of the affine plane at the origin, and let E denote the exceptional divisor.

a). Let L be any line through the origin in $\mathbb{A}^2$. Compute the pullback divisor $\pi^*(L)$. In particular, what is the coefficient of E in $\pi^*(L)$?

b). Let $C = \mathbb{V}(f_d + f_{d+1} + \ldots + f_n)$ be an irreducible curve through the origin in the affine plane, where f_i is a homogeneous polynomial of degree i in x and y and $d < d+1 < \ldots < n$. Describe the pullback divisor $\pi^*(C)$. In particular, what is the coefficient of E in $\pi^*(C)$?

c). Describe the pullback (as a divisor) of any irreducible curve not passing through the origin. In particular, what is the coefficient of E in $\pi^*(C)$?

4. Reduction to Prime Characteristic. a). Show that Spec Z has one dense point and all other points are closed. The dense point is called the *generic point* of Spec $\mathbb{Z}$.

b). Consider the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[X, Y, Z]/(x^3 + y^3 + z^3)$. Think of the induced map on Spectra as a family of schemes parametrized by $\mathbb{Z}$, whose members are the fibers. Compute the fiber over each closed point, and compare to *generic fiber*, that is, the fiber over the dense point.[1]

[1]This is the beginning of a very powerful technique called "reduction to prime characteristic."

c). Most of the fibers are nice and similar to eachother, but there is a closed set of "bad fibers". Find such a closed set and explain how is it bad and/or different from the others (there may be more than one way do to this for an arbitrary family, though in this case, there is one "obvious" thing to look at). Is the generic fiber in the good set or the bad set? Why is it called the generic fiber?

5. Separating Points and Tangent Vectors. A morphism ϕ of varieties *separates points* if it is one-to-one, and separates tangent vectors at p if the induced map $d_p\phi$ of tangent spaces is injective.

a). Which of the following morphisms separates points? Where does each separate tangent vectors? (Assuming the characteristic is not 2 or 3).

$\phi : \mathbb{A}^1 \to \mathbb{A}^2$ sending $t \mapsto (t, t^2)$.

$\phi : \mathbb{A}^1 \to \mathbb{A}^2$, sending $t \mapsto (t^2, t^3)$.

$\phi : \mathbb{A}^1 \to \mathbb{A}^2$, sending $t \mapsto (t^2 - 1, t(t^2 - 1))$. (Hint: p 4 of Shaf).

b). For all three examples above, draw pictures of the image curves and discuss the geometric intuition of separating points and tangent vectors. Note from this exercise that **a bijective regular map need not be an isomorphism!** [2]

c). For the A+ crowd: Prove that a surjective morphism between projective varieties $X \to Y$ is an isomorphism if and only if it separates points, and separates tangent vectors at every point.

6. Varieties over non-algebraically closed fields. a). Consider the inclusion of rings $\mathbb{R}[X] \subset \mathbb{C}[X]$ and the corresponding induced map ϕ :Spec $\mathbb{C}[x] \to$ Spec $\mathbb{R}[X]$. Explicitly describe all the fibers (both algebraically and geometrically).

b). Show that conjugation induces an action of the group of two elements on Spec $\mathbb{C}[x]$. What are the fixed points of this action? Show that the induced quotient space is homeomorphic to Spec $\mathbb{R}[X]$.

c). For the A+ crowd: There is a lot of neat stuff to discover in this direction more generally.

[2] Under some fairly weak assumptions, bijective regular morphisms are isomorphisms. "Zariski's Main theorem" says that if $\phi : V \to W$ is a bijective regular map of affine irreducible varieties over an algebraically closed field of characteristic zero, and W is *normal* (meaning its coordinate ring is integrally closed in its fraction field), then ϕ is an isomorphism. See Hartshorne, page 280.