Math 631: Problem Set 11

Due Monday December 8, 2008

1. Linear Systems on Projective Space. Let $D$ be the divisor $dH$ on $\mathbb{P}^n$ where $d$ is a fixed positive number and $H$ is any hyperplane.

a). Describe the Riemann-Roch space $\mathcal{L}(D)$, and give a formula for its dimension in terms of $d$. Also, describe the complete linear system $|D|$.

b). Explicitly describe the rational map given by the complete linear system $|D|$. What is its locus of indeterminancy? Is $D$ very ample?

c). Now let $d=1$ and consider the linear system of hyperplanes in $\mathbb{P}^n$ passing through a fixed point $p$. Verify that this really is a linear system by finding a vector subspace of some Riemann-Roch space that determines it. Explicitly describe the map given by this linear system. What is its locus of indeterminancy?

d). Again with $d$ arbitrary, consider the sublinear system of $|D|$ of degree $d$ hypersurfaces passing through a point $p$ (for some fixed $p$). Explicitly describe the map given by this linear system. What is its locus of indeterminancy?

2. Linear Systems on Products of Projective Spaces. a) On $\mathbb{P}^1 \times \mathbb{P}^1$, consider the complete linear system of divisors linearly equivalent to the divisor $D = \mathbb{P}^1 \times x$ for some fixed point $x \in \mathbb{P}^1$. Describe the members of this linear system explicitly. Find a basis for $\mathcal{L}(D)$. What is the rational map determined by the complete linear system $|D|$? What is its base locus? Is it very ample?

b). On $\mathbb{P}^1 \times \mathbb{P}^1$, let $D$ be any effective divisor in the class corresponding to $(1,1)$ under the isomorphism from Problem 2 on Homework 10. Compute $\mathcal{L}(D)$ explicitly, and also describe the complete linear system $|D|$, drawing a sketch of how the members look in an affine patch $\mathbb{A}^1 \times \mathbb{A}^1$. Explicitly describe the map given by the complete linear system $|D|$. What familiar map is it? Is $D$ very ample? Base point free?

3. a). Prove that every rational map from $\mathbb{P}^n$ to another projective space is the composition of a Veronese embedding followed by projection from some linear subspace. Of these, which are morphisms?

b). Describe all possible maps from $\mathbb{P}^n \times \mathbb{P}^m$ to projective space (your answer should be a statement along the lines of (a), though of course it is more complicated).

c). Prove that the group $Aut(\mathbb{P}^n)$ of automorphisms of $\mathbb{P}^n$ is the projective linear group $PGL_k(n+1)$.

d). Given an example to show that not all isomorphisms are projective changes of coordinates by finding two curves in $\mathbb{P}^2$ that are isomorphic but not projectively equivalent.

4. Invertible Sheaves. Let $X$ be an irreducible normal variety, $D$ a divisor on $X$. For every open set $U$ of $X$, define $\mathcal{O}_X(D)(U) = \{ f \in k(U)^* | div f + (D \cap U) \geq 0 \} \cup 0$.

a). Show that $\mathcal{O}_X(D)$ defines a sheaf of abelian groups on $X$. 

b). Show that $\mathcal{O}_X(D)$ has the structure of sheaf of $\mathcal{O}_X$-modules. [This means that each $\mathcal{O}_X(D)(U)$ is an $\mathcal{O}_X(U)$-module, compatibly with restriction maps: if $V \subset U$, $\phi \in \mathcal{O}_X(U)$ and $g \in \mathcal{O}_X(D)(U)$, then the restriction of $\phi g$ to $V$ is equal to $\phi|_V g|_V$.]

c). Show that if $D$ is Cartier, then $\mathcal{O}_X(D)$ is a locally free $\mathcal{O}_X$-module of rank one. [This means that $X$ has a cover by open sets $U$ such that each $\mathcal{O}_X(D)|_U \cong \mathcal{O}_X|_U$ as modules over $\mathcal{O}_X(U)$.]

d). Now let $\mathcal{L}$ be any locally free $\mathcal{O}_X$-submodule of the constant sheaf $k(X)$ on $X$. Show that $\mathcal{L}$ has rank one, and that there exists a Cartier divisor $D$ such that $\mathcal{L} = \mathcal{O}_X(D)$. (Such a sheaf $\mathcal{L}$ is called an invertible sheaf on $X$.)

e). A homomorphism of sheaves of $\mathcal{O}_X$-modules $\phi : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ of $\mathcal{O}_X(U)$-modules, compatible with restriction. Show that two invertible sheaves $\mathcal{L}$ and $\mathcal{L'}$ are isomorphic if and only their corresponding divisors $D$ and $D'$ are linearly equivalent. Thus the Picard group can be defined as the group of isomorphism classes of invertible sheaves on $X$. (What is the multiplication on this set?)

5. The tautological Bundle. Let $L \to \mathbb{P}^n$ be the tautological bundle on $\mathbb{P}^n$.

a). Explicitly describe the module of sections of this bundle over each of the open sets in the standard affine open cover of $\mathbb{P}^n$.

b). Prove that this tautological bundle has no non-zero global sections.

c). Find a divisor $D$ on $\mathbb{P}^n$ so that the sheaf of sections of $L$ is isomorphic to $\mathcal{O}_X(D)$.

d). For the A+ crowd: Describe the dual line bundle, its global sections, and a corresponding divisor. Do the same for the tensor powers of $L$ and its dual. What does the operation of tensor correspond to in terms of the divisor (classes) $D$?

6. Genus of Plane Curves. The genus of a smooth projective curve is defined as the dimension of the Riemann-Roch space $\mathcal{L}(K_D)$.

a) Compute the genus of a smooth irreducible curve in $\mathbb{P}^2$ of degree $d$, as a function of $d$.

b). What can you say about the canonical map of $X$ (the one given by the complete linear system $|K_X|$).

7. Ramification and Differentials. Let $X$ be the surface in complex three space $\mathbb{A}^3$ defined by an irreducible polynomial $z^n - f(x, y)$. Assume that $X$ is smooth. Consider the projection $\pi : X \to \mathbb{A}^2$ sending $(x, y, z) \mapsto (x, y)$.

a). Compute the degree and the ramification locus of $\pi$. (The ramification locus is the subset of $\mathbb{A}^2$ where there fail to be exactly degree $\pi$ distinct pre-images under $\pi$.)

b). Describe local generators for the regular differential two-forms on $X$ in a neighborhood of a point $p$ on $X$.

c). Explicitly compute the pullback of an arbitrary differential form on $\mathbb{A}^2$ to $X$.

d). How is the ramification locus described in terms of the behavior of differential forms under pull back?

e). What can go wrong in characteristic $p$?