Math 631: Problem Set 11

Due Monday December 8, 2008

1. Linear Systems on Projective Space. Let D be the divisor dH on $\mathbf{P}^n$ where d is a fixed positive number and H is any hyperplane.

a). Describe the Riemann-Roch space $\mathcal{L}(D)$, and give a formula for its dimension in terms of d. Also, describe the complete linear system $|D|$.

b). Explicitly describe the rational map given by the complete linear system $|D|$. What is its locus of indeterminancy? Is D very ample?

c). Now let $d = 1$ and consider the linear system of hyperplanes in $\mathbf{P}^n$ passing through a fixed point p. Verify that this really is a linear system by finding a vector subspace of some Riemann-Roch space that determines it. Explictly describe the map given by this linear system. What is its locus of indeterminancy?

d). Again with d arbitrary, consider the sublinear system of $|D|$ of degree d hypersurfaces passing through a point p (for some fixed p). Explictly describe the map given by this linear system. What is its locus of indeterminancy?

2. Linear Systems on Products of Projective Spaces. a) On $\mathbf{P}^1 \times \mathbf{P}^1$, consider the complete linear system of divisors linearly equivalent to the divisor $D = \mathbf{P}^1 \times x$ for some fixed point $x \in \mathbf{P}^1$. Describe the members of this linear system explictly. Find a basis for $\mathcal{L}(D)$. What is the rational map determined by the complete linear system $|D|$? What is its base locus? Is is very ample?

b). On $\mathbf{P}^1 \times \mathbf{P}^1$, let D be any effective divisor in the class corresponding to $(1, 1)$ under the isomorphism from Problem 2 on Homework 10. Compute $\mathcal{L}(D)$ explicitly, and also describe the complete linear system $|D|$, drawing a sketch of how the members look in an affine patch $\mathbb{A}^1 \times \mathbb{A}^1$. Explicitly describe the map given by the complete linear system $|D|$. What familiar map is it? Is D very ample? Base point free?

3. a). Prove that every rational map from $\mathbf{P}^n$ to another projective space is the composition of a Veronese embedding followed by projection from some linear subspace. Of these, which are morphisms?

b). Describe all possible maps from $\mathbf{P}^n \times \mathbf{P}^m$ to projective space (your answer should be a statement along the lines of (a), though of course it is more complicated).

c). Prove that the group $Aut(\mathbf{P}^n)$ of automorphisms of $\mathbf{P}^n$ is the projective linear group $PGL_k(n + 1)$.

d). Given an example to show that not all isomorphisms are projective changes of coordinates by finding two curves in $\mathbf{P}^2$ that are isomorphic but not projectively equivalent.

4. Invertible Sheaves. Let X be an irreducible normal variety, D a divisor on X. For every open set U of X, define $\mathcal{O}_X(D)(U) = \{f \in k(U)^* \,|\, div f + (D \cap U) \geq 0\} \cup 0$.

a). Show that $\mathcal{O}_X(D)$ defines a sheaf of abelian groups on X.

b). Show that $\mathcal{O}_X(D)$ has the structure of sheaf of $\mathcal{O}_X$-modules. [This means that each $\mathcal{O}_X(D)(U)$ is an $\mathcal{O}_X(U)$-module, compatibly with restriction maps: if $V \subset U$, $\phi \in \mathcal{O}_X(U)$ and $g \in \mathcal{O}_X(D)(U)$, then the restriction of ϕg to V is equal to $\phi_{|V} g_{|V}$.]

c). Show that if D is Cartier, then $\mathcal{O}_X(D)$ is a locally free $\mathcal{O}_X$-module of rank one. [This means that X has a cover by open sets U such that each $\mathcal{O}_X(D)_{|U} \cong \mathcal{O}_{X|U}$ as modules over $\mathcal{O}_X(U)$.]

d). Now let $\mathcal{L}$ be any locally free $\mathcal{O}_X$-submodule of the constant sheaf $k(X)$ on X. Show that $\mathcal{L}$ has rank one, and that there exists a Cartier divisor D such that $\mathcal{L} = \mathcal{O}_X(D)$. (Such a sheaf $\mathcal{L}$ is called an *invertible* sheaf on X.)

e). A homomorphism of sheaves of $\mathcal{O}_X$-modules $\phi : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ of $\mathcal{O}_X(U)$-modules, compatible with restriction. Show that two invertible sheaves $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic if and only their corresponding divisors D and D' are linearly equivalent. Thus the Picard group can be defined as the group of isomorphism classes of invertible sheaves on X. (What is the multiplication on this set?).

5. The tautological Bundle. Let $L \longrightarrow \mathbf{P}^n$ be the tautological bundle on $\mathbf{P}^n$.

a). Explictly describe the module of sections of this bundle over each of the open sets in the standard affine open cover of $\mathbf{P}^n$.

b). Prove that this tautological bundle has no non-zero global sections.

c). Find a divisor D on $\mathbf{P}^n$ so that the sheaf of sections of L is isomorphic to $\mathcal{O}_X(D)$.

d). For the A+ crowd: Describe the dual line bundle, its global sections, and a corresponding divisor. Do the same for the tensor powers of L and its dual. What does the operation of tensor correspond to in terms of the divisor (classes) D?

6. Genus of Plane Curves. The genus of a smooth projective curve is defined as the dimension of the Rieman-Roch space $\mathcal{L}(K_D)$.

a) Compute the genus of a smooth irreducible curve in $\mathbf{P}^2$ of degree d, as a function of d.

b). What can you say about the canonical map of X (the one given by the complete linear system $|K_X|$).

7. Ramification and Differentials. Let X be the surface in complex three space $\mathbb{A}^3$ defined by an irreducible polynomial $z^n - f(x, y)$. Assume that X is smooth. Consider the projection $\pi : X \longrightarrow \mathbb{A}^2$ sending $(x, y, z) \mapsto (x, y)$.

a). Compute the degree and the ramification locus of π. (The ramification locus is the subset of $\mathbb{A}^2$ where there fail to be exactly degree π distinct pre-images under π.)

b). Describe local generators for the regular differential two-forms on X in a neighborhood of a point p on X.

c). Explicitly compute the pullback of an arbitrary differential form on $\mathbb{A}^2$ to X.

d). How is the ramification locus described in terms of the behavior of differential forms under pull back?

e). What can go wrong in characteristic p?