

# Math 631: Algebraic Geometry: Homework Set 2.

Due Friday September 19, 2008

Assume the ground field  $k$  is algebraically closed, wherever necessary (or convenient).

**1.** a). Prove that every morphism (regular map) of affine algebraic sets is continuous in the Zariski topology.

b). Prove or find a counterexample to the converse statement.

**2.** Let  $F : V \rightarrow W$  be a morphism of affine algebraic varieties and let  $F^* : k[W] \rightarrow k[V]$  be the induced map of coordinate rings.

a). Prove that  $F$  is dominant if and only if  $F^*$  is injective. (A map is dominant if its image is dense in the target space.)

b) Prove that  $F$  is a closed embedding if and only if  $F^*$  is surjective. (A closed embedding is an isomorphism onto a closed set of the target space.)

**3.** Let  $V$  be an affine algebraic variety (affine algebraic subset of  $\mathbb{A}^n$ ), and let  $k[V]$  be its coordinate ring.

a). For each  $f \in k[V]$ , let  $\mathcal{U}_f$  be the set of points  $P \in V$  such that  $f(P) \neq 0$ . Prove that the collection of all such sets  $\mathcal{U}_f$  forms a basis for the Zariski topology on  $V$ . The set  $\mathcal{U}_f$  is called a basic (or principle) open set.

b). Prove that  $V$  is compact (every open cover has a finite subcover).

c). Is  $V$  Hausdorff? Describe all Hausdorff algebraic varieties.

**4.** With notation as in Exercise 3, consider the map

$$\begin{aligned} \mathcal{U}_f &\rightarrow \mathbb{A}^n \times \mathbb{A}^1 \\ P &\mapsto \left(P, \frac{1}{f(P)}\right). \end{aligned}$$

a). Show that this map defines a bijection from  $\mathcal{U}_f$  onto an algebraic subset of  $\mathbb{A}^{n+1} = \mathbb{A}^n \times \mathbb{A}^1$ .

b). Show in fact that this map defines a homeomorphism of  $\mathcal{U}_f$  with its image. (The topology on  $\mathcal{U}_f$  is the subspace topology induced by the Zariski topology on  $V$ .)

c) Find the coordinate ring of the image and show that it is isomorphic to  $k[V][\frac{1}{f}]$ .

d). Conclude that the Zariski topology on any open subset of  $V$  (considered with the subspace topology) has a basis of sets that can be identified with affine algebraic varieties.

5. a). At what points of the “circle” given by  $\mathbf{V}(x^2 + y^2 - 1)$  in  $\mathbb{A}^2$  is the rational function  $(1 - y)/x$  regular? (Your answer might depend on the characteristic.)

b). At what points of the “nodal curve” given by  $\mathbf{V}(y^2 - x^2 - x^3)$  in  $\mathbb{A}^2$  is the rational function  $y/x$  regular?

**6. Linear Subvarieties of projective space.** Fix a finite-dimensional vector space  $V$ , so the points in  $\mathbf{P}(V)$  are the one-dimensional subspaces of  $V$ . For any subvector space  $W$ , the projective space  $\mathbf{P}(W)$  is called a *linear subspace* of  $\mathbf{P}(V)$ , or a  $d$ -plane in  $\mathbf{P}(V)$  if  $W$  has dimension  $d + 1$ . A 1-plane is also called a line.

a). Prove that  $\mathbf{P}(W)$  is a Zariski closed subset of  $\mathbf{P}(V)$ , and that it is the intersection of  $c$  hyperplanes in  $\mathbf{P}(V)$ , where  $c$  is the codimension of  $\mathbf{P}(W)$ .

b). Prove that if  $\mathbf{P}(W_1)$  and  $\mathbf{P}(W_2)$  are linear subspaces of codimension  $c_1$  and  $c_2$  respectively, then  $\mathbf{P}(W_1) \cap \mathbf{P}(W_2)$  is a linear subspace of codimension at most  $c_1 + c_2$ .

**7. Abstract Algebraic Varieties.** Let  $V$  be an irreducible affine algebraic set, with function field  $k(V)$ . For each open set  $U \subset V$ , let  $\mathcal{O}_V(U)$  be the set of all rational functions  $\phi \in k(V)$  that are regular at each point  $P \in U$ .

a). Look up the definition of a sheaf on page 61 of Hartshorne. Prove that the assignment  $U \mapsto \mathcal{O}_V(U)$  makes  $\mathcal{O}_V$  into a sheaf of rings on  $V$ .

b). A *ringed space*  $(V, \mathcal{O}_V)$  is a topological space  $V$  together with a sheaf of rings on it. Deduce that every irreducible affine algebraic set has the structure of a ringed space.<sup>1</sup> Explain also how topological spaces with continuous functions, differentiable manifolds with smooth functions, and complex manifolds<sup>2</sup> with holomorphic functions are naturally ringed spaces as well.

c). Show that for any open set  $U$  of  $V$ , there is a naturally induced ringed space structure  $(U, \mathcal{O}_{V|U})$  on  $U$ .

d). Describe a notion of isomorphism of ringed spaces.

e). An abstract algebraic variety over  $k$  can be defined as a ringed space  $(V, \mathcal{O}_V)$  which admits an open cover  $\{U_i\}_{i \in I}$  such that each  $(U_i, \mathcal{O}_{V|U_i})$  is isomorphic to the ringed space of some irreducible affine algebraic set over  $k$ .<sup>3</sup> Show that every open subset of an affine algebraic set has the structure of an abstract variety.

g). A abstract variety is said to be *affine* if it is isomorphic (as ringed space) to an affine algebraic variety. Show that  $GL_n$  has the structure of an affine variety even though it is not an affine algebraic subset of  $\mathbb{A}^{n^2}$ .

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<sup>1</sup>It is only slightly more technical algebraically to figure out the case where  $V$  has multiple components. You don't have to do this here, but might want to think about it.

<sup>2</sup>or just  $\mathbb{C}^n$  if you aren't familiar with complex manifolds

<sup>3</sup>Some authors would call this a prevariety, and require an additional assumption on the topological space called "separatedness" (which amounts to something like the Hausdorff axiom for manifolds) in order for it to be a variety. In fact, a variety over the complex numbers is separated if and only if it is Hausdorff when considered with the euclidean topology. Note that all complex varieties have a Euclidean topology since they locally look like subsets of complex space. We won't go into separatedness in Math 631, because all the examples we study will automatically satisfy this condition, as do essentially all the examples that come up for the algebraic geometer on the street.