Math 631: Problem Set 3

Due Friday September 26, 2008

1. Find the domain of definition of the rational function $\phi = \frac{x}{z}$ on the affine algebraic subset of $\mathbb{A}^3$ defined by $z^2 - xy$.

2. Projective Nullstellensatz. Recall that an ideal a polynomial ring $k[X_0, \ldots, X_n]$ is said to be *homogeneous* if it can be generated by homogeneous elements.

a). Prove that I is homogeneous if and only if, for all $f \in I$, each homogeneous component of f is in I. (For example, the homogenous components of the polynomial $xy + z^2 - \pi z^{17}$ are the homogeneous polynomials $xy + z^2$ and $-\pi z^{17}$.)

b). Let I and J be homogeneous ideals of $k[X_0, \ldots, X_n]$. Show that $I + J$, IJ, $I \cap J$, and $\sqrt{I}$ are all homogeneous.

c). State (be careful!) a projective version of Hilbert's Nullenstellensatz and show how it follows from the affine version.

3. Projective Equivalence. Two projective varieties X and Y in $\mathbf{P}^n$ are *projectively equivalent* if there is a linear changes of coordinates on $\mathbf{P}^n$ that restricts to an isomorphism $X \to Y$.

a). Assuming, the ground field does not have characteristic two (and is algebraically closed), prove that any two non-singular quadrics in $\mathbf{P}^n$ are projectively equivalent. (A quadric is non-singular if the corresponding homogeneous polynomial of degree two is non-singular considered as a quadratic form on k^{n+1}, ie, if the matrix of the associated bi-linear form is invertible.)

b). Give an example of two subvarieties of $\mathbb{P}^2$ that isomorphic but not projectively equivalent.

4. Prove that a quasi-projective variety in $\mathbf{P}^n$ is irreducible if and only if it is the intersection of an open set in $\mathbf{P}^n$ with an irreducible projective variety in $\mathbf{P}^n$.

5. Projection from a linear space. Let Λ and L be two disjoint linear subvarieties in $\mathbf{P}(V)$, and assume that $\dim \Lambda + \dim L = \dim P(V) - 1$.

a). For any point $x \in \mathbf{P}(V) - \Lambda$, show that there is a unique linear space of dimension $\dim \Lambda + 1$ containing both Λ and x. Show that this linear space intersects L in precisely one point, call it $\pi_{\Lambda,L}(x)$.

b). Express $\pi_{\Lambda,L}$ explicitly in coordinates[1] and prove that it is regular on $\mathbf{P}^n - \Lambda$.

c). Show that π_Λ is the composition of a sequence of projections from points.

d). Let $X \subset \mathbf{P}(V)$ be any quasiprojective variety disjoint from Λ. Show that (up to projective equivalence, the image of X under $\pi_{\Lambda,L}$ does not depend on the choice of L. That is, if Λ is fixed, show there exists an isomorphism $L \to L'$ given by linear polynomials taking the subvariety $\pi_{\Lambda,L}(X)$ isomorphically to $\pi_{\Lambda,L'}(X)$. For this reason, we usually just say π_Λ is the projection from Λ.

e). Now fix L and consider the different projections $\pi_{\Lambda,L}$ as Λ varies. Does the image $\pi_{\Lambda,L}(X)$ depend on the choice of Λ? Prove or give a counterexample.

[1] Please *choose* coordinates wisely!

6. The Twisted Cubic. The "twisted cubic" is, by definition, the image of the Veronese map ν_3 of $\mathbf{P}^1$ into $\mathbf{P}^3$.

a). Describe ν in affine coordinates on some (well-chosen) affine patch, and describe the corresponding map on coordinate rings.

b). Show that the twisted cubic (call it C) lies on the three "quadric surfaces," $Q_0 = \mathbb{V}(X_0X_2 - X_1^2)$, $Q_1 = \mathbb{V}(X_0X_3 - X_1X_2)$, and $Q_2 = \mathbb{V}(X_1X_3 - X_2^2)$; and that $C = Q_0 \cap Q_1 \cap Q_2$.

c). Show that any two of the quadrics Q_i intersect in the union of C and a line.

d). Show that the twisted cubic is the intersection of two hypersurfaces in $\mathbf{P}^3$. Do the equations you found generate the full radical ideal of homogenous polynomials vanishing on C?

e). Show that the twisted cubic is (up to projective equivalence) the intersection of the Segre 3-fold[2] in $\mathbf{P}^5$ with a 3-plane in $\mathbf{P}^5$.

7. The local ring at a point. Let V be an irreducible variety. Fix a point $x \in V$, and consider the subset $O_{x,V}$ of $k(V)$ consisting of those rational functions on V regular at x.

a). Prove that $O_{x,V}$ is a ring.

b).Prove that the set of rational functions in $O_{x,V}$ which vanish at x forms an ideal $\mathfrak{m}_x$ of $O_{x,V}$. Furthermore, show that $\mathfrak{m}_x$ is the unique maximal ideal of $O_{x,V}$, so that (by definition) $O_{x,V}$ is a local ring.

c). Suppose the rational function ϕ is regular at x, so that $\phi \in O_{x,V}$. Prove that the value of ϕ at x is equal to the residue class of ϕ modulo $\mathfrak{m}_x$.

d). Show that $O_{x,V}$ can be expressed as the direct limit, over all open sets U of V containing x, of the rings $O_V(U)$, where the maps in the direct limit system are given by restriction.

e.) Look up the definition of the stalk of a sheaf at a point in Hartshorne, and conclude that the stalk of $\mathcal{O}_V$ at a point $x \in V$ is precisely $O_{x,V}$. Look up the definition of a locally ringed space in Hartshorne, and prove that every irreducible variety has a natural structure of a locally ringed space.

[2]Recall the Segre 3-fold is the image of $\mathbf{P}^1 \times \mathbf{P}^2$ under the Segre map Σ_{12} to $\mathbf{P}^5$.